Homework 1 Solutions

Sam Mundy

Exercise 1. List all subsets of the 3-element set $A = \{1, 2, 3\}$. How many subsets does a set with n elements have? How many of these subsets have at most two elements?

Solution. The set A has $8 = 2^3$ subsets, namely

$$\emptyset,\ \{1\},\ \{2\},\ \{3\},\ \{1,2\},\ \{1,3\},\ \{2,3\},\ \{1,2,3\}.$$

In general, a set S with n elements has 2^n subsets. Intuitively, for every element in S, you get two choices when constructing a subset $T \subseteq S$: You can choose whether or not this element will appear in T. These choices multiply, and so S has $2^n = 2 \cdot 2 \cdots 2$ (n times) subsets.

However, there are a lot of ways you can prove this rigorously. One way is to use the principal of mathematical induction (details left to the interested reader.) Another way, which many of you followed, is to note that there are $\binom{n}{k}$ k-element subsets of an n-element set. Summing over k means there are

$$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$$

subsets total.

For the last part, to count subsets with at most 2 elements, we simply use the method just described. This gives that there are

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} = 1 + n + \frac{n(n-1)}{2}$$

subsets of size at most 2.

Exercise 2. Simplify descriptions of the following sets. These sets depend on subsets A, B of a universal set X, so that $A' = X \backslash A$, and so on.
(a) $A' \cup A$, $\quad A' \cap A$, $\quad (A' \cup A')' \cup (A' \cap A)$.
(b) $(A \cap B) \cup (A \cup B)$, $\quad (A \cup B') \cap (A' \cap B)$,
$(A \cap B) \backslash B$, $\quad (A \cup B) \backslash B$, $\quad (A \cap B) \cup (A \backslash B)$.

Solution. For (a), we have

$$A' \cup A = X$$

and

$$A' \cap A = \emptyset.$$

For the last one, we have
$$(A' \cup A')' = (A')' = A,$$
hence
$$(A' \cup A')' \cup (A' \cap A) = A \cup \emptyset = A.$$

For the first expression in (b), we know that $A \cap B$ is a subset of $A \cup B$, because is it a subset of A. Thus,
$$(A \cap B) \cup (A \cup B) = A \cup B.$$
For the second, we use distributivity:
$$(A \cup B') \cap (A' \cup B) = (A \cap (A' \cap B)) \cup (B' \cap (A' \cap B)).$$
By associativity and commutativity of intersections, this equals
$$((A \cap A') \cap B) \cup (A' \cap (B' \cap B)) = \emptyset \cup \emptyset = \emptyset.$$
For the third expression, since $(A \cap B)$ is a subset of B, we have
$$(A \cap B)\backslash B = \emptyset.$$
For the fourth, we can distribute the difference over the union:
$$(A \cup B)\backslash B = (A\backslash B) \cup (B\backslash B) = A\backslash B.$$
Finally, for the last expression, we have
$$(A \cap B) \cup (A\backslash B) = (A \cap B) \cup (A \cap B') = A \cap (B \cup B') = A \cap X = A.$$

Exercise 4. (a) Consider sets $A = \{a, b\}$ and $B = \{1, 2, 3\}$. How many injective maps are there from A to B. Give an example of such a map. How many injective maps are there from B to A?

(b) Suppose $f : A \to A$ is injective and A is a finite set. Prove that f is bijective. Give an example of an infinite set B and an injective map $f : B \to B$ which is not surjective.

Solution. (a) By a general formula that can be found in Prof. Gallagher's notes, there are $3 \cdot 2$ injective maps from A to B. (In general, the formula is *not* simply the product of the sizes of both sets.) It is also possible to list them. One such function f is the one that sends a to 1 and b to 2.

On the other hand, there are no injective maps from B to A simply because B has more elements than A.

(b) Since f is injective, it is sufficient to show that it is also surjective. This will show that it is bijective. To show surjectivity, we note that by the injectivity of f, we have $|A| = |f(A)|$. (Here, $f(A)$ denotes the image of f.) But we know that $f(A) \subseteq A$, so this shows that actually $A = f(A)$. This says exactly that f is surjective, and we are done. (Note that there is no need to prove this by contradiction.)

An example of an injective but not surjective map $g : B \to B$, with B infinite, is given as follows. Let $B = \mathbb{N}$, and define g by $g(n) = n + 1$. Then g is injective, but 1 is not in the image (for me the first natural number is 1, not 0.) Other popular answers were $B = \mathbb{N}$ and $g(n) = 2n$, and $B = \mathbb{R}$ and $g(x) = e^x$, which are all correct.

Exercise 5. (a) Show that, for any set A, there is exactly one map f from the empty set $\emptyset$ to A. When is A injective? Surjective?

(b) Describe all maps from a set A to the empty set $\emptyset$ (the answer will depend on A).

Solution. (a) This map $f : \emptyset \to A$ exists for vacuous reasons. In more detail, let us go back to the rigorous definition of a function. For sets A and B, a function $A \to B$ is a subset $f \subseteq A \times B$ such that the following condition holds: For all $a \in A$, there is a unique ordered pair $(a, b) \in f$ (usually written "$f(a) = b$") with a as the first entry. Now in our case, we are looking for a function $f : \emptyset \to A$. Rigorously speaking, this is a subset of $\emptyset \times A$ satisfying the property above. But $\emptyset \times A = \emptyset$. So we take our function $f : \emptyset \to A$ to be the only subset of $\emptyset \times A = \emptyset$, namely $\emptyset$ itself. Then we need to check the property above, that is, we need to show that for any element $x \in \emptyset$, there is a unique ordered pair (x, a) with $a \in A$ such that $f(x) = a$. But this is vacuously true since there is no $x \in \emptyset$! This function is unique since there are no other subsets of $\emptyset \times A = \emptyset$.

This map g is injective again for vacuous reasons, and it is only surjective when $A = \emptyset$, again for vacuous reasons.

(b) There are two cases. First assume that A is nonempty. Then a function $f : A \to \emptyset$ would have to be a rule that assigns to every element of A an element of $\emptyset$. There are no elements of $\emptyset$, but by assumption, there are elements of A, so no such f can exist.

On the other hand, if $A = \emptyset$, then by (a) above, there is exactly one function $f : \emptyset \to \emptyset$, and we have already described it.

Exercise 6. We saw in class that, for sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Prove that, for sets A, B, C,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Solution. By the formula from class,

$$|A \cup B \cup C| = |A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)|.$$

By the same formula applied to $B \cup C$, this is

$$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)|.$$

Now write $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Then invoking the formula from class one more time, we get

$$|(A \cap B) \cup (A \cap C)| = |A \cap B| + |A \cap C| - |A \cap B \cap A \cap C|.$$

Of course, $A \cap B \cap A \cap C = A \cap B \cap C$. Thus, substituting, we get

$$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|.$$

This is the desired formula.