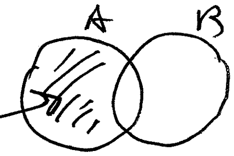
① $(A \cup B) \setminus B = A \setminus B$ for any two sets A and B

True LHS and RHS are both



② $A \setminus (A \setminus B) = B$ for any two sets A and B

False Take $A = \{a\}$, $B = \{b\}$. Then LHS (left hand side) is $\emptyset$, while RHS $= \{b\}$.

③ $75 \equiv -10 \pmod{17} \iff 75 + 10 \equiv 0 \pmod{17} \iff$

$85 \equiv 0 \pmod{17}$. True, since $85 = 17 \cdot 5$

④ $n \sim m$ iff $5 \mid n - m$ True

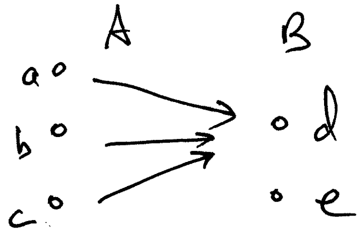
This is indeed an equivalence relation on $\mathbb{Z}$.

The equivalence classes are residues modulo 5

⑤ $\gcd(n,m) = \gcd(m,n)$ for any natural numbers n, m

True

(2) $A = \{a, b, c\}$, $B = \{d, e\}$



We can make all elements of A go to the same element of B. For insitance,

$$f(a) = f(b) = f(c) = d$$

Then f is <u>not</u> surjective, since e is not in the image of f.

(3) A vector $\vec{v} \in \mathbb{R}^2$.

We know that addition of vectors is associative.

$$(\vec{v}_1 + \vec{v}_2) + \vec{v}_3 = \vec{v}_1 + (\vec{v}_2 + \vec{v}_3).$$

Vector $\vec{0}$ has the property that $\vec{0} + \vec{v} = \vec{v} = \vec{v} + \vec{0}$ for any vector $\vec{v} \in \mathbb{R}^2$.

For any vector $\vec{v}$ vector $-\vec{v}$ has the property

$$\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}.$$

Therefore, the set of vectors in $\mathbb{R}^2$ constitute a group under addition. The unit element is the vector $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The inverse of $\vec{v}$ is $-\vec{v}$. If $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ then $-\vec{v} = \begin{pmatrix} -a \\ -b \end{pmatrix}$.

(4) First list all residues modulo 10

$0, 1, 2, 3, 4, 5, 6, 7, 8, 9.$

Next remove all those not coprime to 10 (those that share a divisor with 10, prime divisors are 2 and 5).

$\cancel{0}, 1, \cancel{2}, 3, \cancel{4}, \cancel{5}, \cancel{6}, 7, \cancel{8}, 9$ removed all even residues mod 10

removed 5

We have $\mathbb{Z}_{10}^* = \{1, 3, 7, 9\}$

The order is four, $|\mathbb{Z}_{10}^*| = 4.$

order of 1 is 1

order of 3: $3^1 \equiv 3, \; 3^2 \equiv 9 \equiv -1, \; 3^3 \equiv (-1)3 \equiv 7, \; 3^4 \equiv 1$

residue 3 has order 4

order of 7: $7^2 \equiv 49 \equiv -1 \pmod{10} \quad 7^3 \equiv -7 \equiv 3 \pmod{10}, \; 7^4 \equiv 1 \pmod{10}$

residue 7 has order 4

order of 9: $9^2 \equiv (-1)^2 \equiv 1 \pmod{10}$

residue 9 has order 2

Mult. table:

	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

The group is cyclic.

3 is a generator,

7 is a generator

$\mathbb{Z}_{10}^* = \langle 3 \rangle = \langle 7 \rangle$

5(a) A subgroup H of a group G is a subset of G that

(a) contains the unit element e,

(b) contains h^{-1} for any $h \in H$,

(c) contains ~~h~~ $h_1 h_2$ for any $h_1, h_2 \in H$.

(b) $H = \{2^k : k \in \mathbb{Z}\}$.

(a) H contains the unit element 1, since $1 = 2^0$ (take $k=0$)

(b) $(2^k)^{-1} = 2^{-k}$ so 2^{-k} is in H.

(c) ~~2^k~~, given $2^k, 2^m \in H$

$$2^k \cdot 2^m = 2^{k+m} \in H, \text{ since } k+m \in \mathbb{Z}.$$

Therefore, H satisfies axioms of a subgroup. Thus, H is a subgroup of $\mathbb{Q}^*$

6. $ab^2a = 1$. Multiply both sides by a^{-1} on the left

$a^{-1}ab^2a = a^{-1}1 \Rightarrow b^2a = a^{-1}$.

Multiply both sides by a^{-1} on the right

$b^2 a a^{-1} = a^{-1}a^{-1}$

$\underline{b^2 = a^{-2}}$

D.

Extra Credit

We can take the subgroup generated by a third root of unity $e^{\frac{2\pi i}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

This subgroup is $\left\{1, e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}\right\}$