§5. Inverse Maps, Partitions

DEF Let $f: X \to Y$ be a bijective map. For $y \in Y$, there is exactly one*) $x \in X$ for which $f(x) = y$. We denote this x by $f^{-1}(y)$. This defines the <u>inverse map</u> $f^{-1}: Y \to X$. A shorter description: For bijective $f: X \to Y$,

$$\boxed{f^{-1}(y) = x \iff y = f(x)} \quad \text{for } x \in X, y \in Y.$$

We emphasize that the inverse map f^{-1} is only defined for bijective f.

THM A. (1) For each set X, the identity map $1_X: X \to X$, defined by $1_X(x) = x$ for all $x \in X$, is bijective, and

$$1_X^{-1} = 1_X.$$

(2) If $f: X \to Y$ is bijective, so is $f^{-1}: Y \to X$, and

$$(f^{-1})^{-1} = f. \qquad \begin{tikzcd} X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, dashed, "f^{-1}"] \end{tikzcd}$$

(3) If $f: X \to Y$ and $g: Y \to Z$ are bijective, so is $gf: X \to Z$, and

$$(gf)^{-1} = f^{-1}g^{-1}$$

$$\begin{tikzcd} & Y \arrow[dl, shift left, dashed] \arrow[dr, shift left, "g"] & \\ X \arrow[ur, shift left, "f"] \arrow[rr, shift left, "gf"] & & Z \arrow[ll, shift left, dashed, "(gf)^{-1}"] \arrow[ul, shift left, dashed] \end{tikzcd}$$

"Put on socks, then shoes; take off shoes, then socks."

(1) Clear, on reflection.

(2) For $x \in X$, $f^{-1}(y) = x$ with $y = f(x)$, showing f^{-1} is surjective. If $f^{-1}(y_1) = f^{-1}(y_2) = x$, say, then $y_1 = f(x)$ and $y_2 = f(x)$ so $y_1 = y_2$, showing f^{-1} is injective. Thus f^{-1} is bijective. Rewriting $\Box$,

$$f(x) = y \iff x = f^{-1}(y),$$

showing $f = (f^{-1})^{-1}$.

*) At least one $x \in X$ because f is surjective, at most one $x \in X$ because f is injective.

(3) That gf is bijective follows from THM D(3) on 4.4. For $x \in X, z \in Z$,

$$\begin{aligned}
(gf)^{-1}(z) = x &\iff z = (gf)(x) && \text{(def of inverse)}\\
&\iff z = g(f(x)) && \text{(def of composite)}\\
&\iff g^{-1}(z) = f(x) && \text{(def of inverse)}\\
&\iff f^{-1}(g^{-1}(z)) = x && \text{(def of inverse)}\\
&\iff (f^{-1}g^{-1})(z) = x. && \text{(def of composite)}
\end{aligned}$$

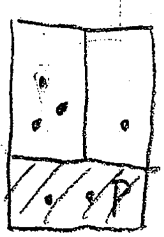
Thus $(gf)^{-1}(z) = (f^{-1}g^{-1})(z)$ for all z, i.e. $(gf)^{-1}$ and $f^{-1}g^{-1}$ are the same map.

EXERCISE 1. Prove that for each bijective $f: X \to Y$,

(a) $f^{-1}f = 1_X$ and $ff^{-1} = 1_Y$;

(b) $f = f1_X$ and $f = 1_Y f$.

DEF Let X be a set. A <u>partition</u> of X is any set $\mathcal{P}$ of <u>disjoint</u>, <u>nonempty</u> subsets P of X which <u>covers</u> X. The P's are the <u>parts</u> of $\mathcal{P}$. In more detail, $\mathcal{P}$ is a partition of X if $\mathcal{P} \subset \mathcal{S}(X)$, and



X

(i) $P_1 \cap P_2 = \phi$ for $P_1, P_2 \in \mathcal{P}$ with $P_1 \neq P_2$ "distinct parts are disjoint"

(ii) $P \neq \phi$ for $P \in \mathcal{P}$ "no part is empty"

(iii) $\bigcup_{P \in \mathcal{P}} P = X$ "$\mathcal{P}$ covers X"

EXAMPLE Each integer is either even (divisible by 2) or odd, ~~but not both~~. This gives a partition of $\mathbb{Z}$ with two parts. More generally, for each $d \in \mathbb{N}$ and $r \in \mathbb{Z}$, let $P_r = \{n \in \mathbb{Z} : n = dq + r \text{ for some } q \in \mathbb{Z}\}$. Then the <u>residue classes</u> $P_0, P_1, \dots, P_{d-1}$ are the parts of a partition of $\mathbb{Z}$ with d parts.

EXERCISE 2. Find all partitions of ϕ, of $\{1\}$, of $\{1,2\}$, and of $\{1,2,3\}$.
Hint: ϕ and $\{1\}$ each have only one partition, and $\{1,2,3\}$ has five.

THM B. (1) For each partition $\mathcal{P}$ of X, define a map $s: X \to \mathcal{P}$ by

$$s(x) = P \iff x \in P, \quad \text{for } x \in X, P \in \mathcal{P}.$$

Then s is surjective. It is the "canonical*) surjection of X to $\mathcal{P}$."
(2). For each subset $B \subset Y$, define a map $i: B \to Y$ by

$$i(y) = y \quad \text{for all } y \in B.$$

Then i is injective. It is the "canonical injection of B to Y."

There is a subtle difference between this i and 1_B, since $i: B \to Y$ and $1_B: B \to B$.

Proof (1). For each $P \in \mathcal{P}$, since $P \neq \phi$ there is at least one $x \in P$, i.e. at least one $x \in X$ with $s(x) = P$. Therefore s is surjective.
(2) If $y_1, y_2 \in B$ and $i(y_1) = i(y_2)$, then $y_1 = i(y_1) = i(y_2) = y_2$, so $y_1 = y_2$.
Therefore i is injective.

THM C (1) Let X, Y be sets, and $f: X \to Y$ any map. Then

$$\{ f^{-1}\{y\} : y \in fX \}$$

is a partition of X, which we denote by X/f. (2) Define a map $b: fX \to X/f$ by

$$b(y) = f^{-1}\{y\} \quad \text{for } y \in fX.$$

Then b is bijective. It is the "canonical bijection of fX to X/f."

Proof. (1) If $y_1, y_2 \in fX$ and $y_1 \neq y_2$ then $f^{-1}\{y_1\} \cap f^{-1}\{y_2\} = \phi$. For $y \in fX$, we have $y = f(x)$ for some $x \in X$, so this $x \in f^{-1}\{y\}$, so $f^{-1}\{y\} \neq \phi$. Finally $x \in f^{-1}\{f(x)\}$ shows that the $f^{-1}\{y\}$'s cover X. (2) As above, $y_1 \neq y_2 \Rightarrow f^{-1}(y_1) \neq f^{-1}(y_2)$ so b

*) Canonical, in mathematics, means standard, i.e. always defined in the same way.

is injective. By definition of b, each part $f^{-1}\{y\}$ of X/f is the value of b at the corresponding $y \in fX$. Thus b is surjective.

THM D. Let X, Y be any sets and $f: X \to Y$ any map. Let

$$s: X \to X/f$$
$$i: fX \to Y$$
$$b: fX \to X/f$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \\ X/f \arrow[r, "b^{-1}"'] & fX \arrow[u, "i"'] \end{tikzcd}$$

be the canonical surjection, injection, bijection defined in Theorems B & C. Then

$$f = i b^{-1} s.$$

Proof. For $x \in X$, let $y = f(x)$. Then $y \in fX$ and $x \in f^{-1}\{y\}$, so $s(x) = f^{-1}\{y\}$.
Since $b(y) = f^{-1}\{y\}$, we have $y = b^{-1}(f^{-1}\{y\}) = b^{-1}(s(x)) = (b^{-1}s)(x)$.
Combining the above,

$$(i b^{-1} s)(x) = i((b^{-1} s)(x)) = i(y) = y = f(x).$$

Because $ib^{-1}s$ & f are maps from X to Y which agree for each $x \in X$, they are equal.

THM E. (1) If Y is finite and $B \subset Y$, then $\#B \leq \#Y$, with $= \Leftrightarrow B = Y$

(2) If X is finite and $\mathcal{P}$ is a partition of X, then $\#\mathcal{P} \leq \#X$, with $= \Leftrightarrow \#P = 1$ for each $P \in \mathcal{P}$.

Proof. (1) Since $Y = B + B'$, $\#Y = \#B + \#B' \geq \#B$ with $= \Leftrightarrow \#B' = 0 \Leftrightarrow B = Y$.

(2) Since $X = \sum_{P \in \mathcal{P}} P$ (disjoint union), $\#X = \sum_{P \in \mathcal{P}} \#P \geq \sum_{P \in \mathcal{P}} 1 = \#\mathcal{P}$, with $= \Leftrightarrow$ each $\#P = 1$.

THM F (1) If $f: X \to Y$ is injective, and Y finite, then $\#X \leq \#Y$ with $= \Leftrightarrow f$ bijective.

(2) If $f: X \to Y$ is surjective, and X finite, then $\#Y \leq \#X$, with $= \Leftrightarrow f$ bijective.

EXERCISE 3. Using THM E, prove THM F.

NOTATION For all nonnegative integers n & k, the number of subsets of size k in any given set of size n is denoted by $\binom{n}{k}$ a.k.a. "n choose k".
For all nonnegative integers n, denote by p_n the number of different partitions of any given set of size n, and put $P(x)=\sum_{n=0}^{\infty} p_n \frac{x^n}{n!}$ for $x\in\mathbb{R}$.

EXERCISE Show that $p_0=1$, $p_1=1$, $p_2=2$, $p_3=5$.

THM F For $0\le k\le n$,

$$\boxed{\binom{n}{k}=\frac{n!}{k!(n-k)!}} \quad \text{with} \begin{cases} 0!=1 \\ k!=k(k-1)! \text{ for } k\in\mathbb{N} \end{cases}$$

Proof. Clearly $\binom{n}{k}=0$ for $k>n$ and $\binom{n}{k}=1$ for $k=0$ and $k=n$.
For $0<k<n$, a subset of size k in $\{1,\cdots,n\}$ is either contained in $\{1,\cdots,n-1\}$ or consists of n together with a subset of size $k-1$ in $\{1,\cdots,n-1\}$, so

$$\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1} \overset{(=)}{=} \frac{(n-1)!}{k!(n-k-1)!}+\frac{(n-1)!}{(k-1)!(n-k)!} \overset{(\equiv)}{=} \frac{n!}{k!(n-k)!},$$

the step $(=)$ by induction and the step $(\equiv)$ by $\frac{1}{k}+\frac{1}{n-k}=\frac{n}{k(n-k)}$.

THM G. (1) For $n\in\mathbb{N}$, $p_n=\sum_{k=1}^{n}\binom{n-1}{k-1}p_{n-k}$. (2) For $x\in\mathbb{R}$, $P(x)=e^{(e^x-1)}$.

(formal argument, not worrying about convergence)

Proof. (1) In any partition of $\{1,\cdots,n\}$, n belongs to a part of size $k\in\{1,\cdots,n\}$. The number of choices for this part is the number of subsets of size $k-1$ in $\{1,\cdots,n-1\}$, and the number of ways of choosing the other parts is p_{n-k}.

(2) $$P'(x)=\sum_{n=1}^{\infty}p_n\frac{x^{n-1}}{(n-1)!}=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{x^{k-1}}{(k-1)!}p_{n-k}\frac{x^{n-k}}{(n-k)!}=\left(\sum_{l=0}^{\infty}\frac{x^l}{l!}\right)\left(\sum_{m=0}^{\infty}p_m\frac{x^m}{m!}\right)=e^xP(x),$$

so $(P(x)e^{-e^x})'=P'(x)e^{-e^x}+P(x)(-e^x)e^{-e^x}=0$, giving $P(x)e^{-e^x}\overset{(=)}{=}\text{const.}=e^{-1}$, as one sees by plugging in $x=0$ in $(=)$, using $P(0)=1$ & $e^0=1$.