

4. Algebra of Maps

5.1

DEF. Given sets X and Y , a function i.e. map $f: X \rightarrow Y$ is any rule which associates to each $x \in X$ some unique $y \in Y$ which (usually) depends on x , and is denoted by $f(x)$.

EXAMPLE For $X = \mathbb{R}$ and $Y = \mathbb{R}$, $f(x) = x^2$ defines a map $f: \mathbb{R} \rightarrow \mathbb{R}$.

For any X and Y and $c \in Y$, $f(x) = c$ for all $x \in X$ defines a constant map.

DEF Given sets X and Y , and a map $f: X \rightarrow Y$,

(1) For each $A \subset X$, the image fA of A by f is the subset of Y defined by

$$y \in fA \iff y = f(x) \text{ for some } x \in A;$$

(2) For each $B \subset Y$, the inverse image $f^{-1}B$ of B by f is the subset of X defined by

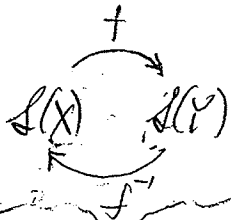
$$x \in f^{-1}B \iff f(x) \in B.$$

REMARK Recall the notation $\mathcal{S}(X)$ for the set of all subsets of X .

In the above definition we have defined, for each map $f: X \rightarrow Y$ two new maps, one from $\mathcal{S}(X)$ to $\mathcal{S}(Y)$, taking $A \subset X$ to $fA \subset Y$, the other from $\mathcal{S}(Y)$ to $\mathcal{S}(X)$, taking $B \subset Y$ to $f^{-1}B \subset X$.

We will denote the first of these by f (same notation but different meaning from the $f: X \rightarrow Y$ we started with), and the second by f^{-1} (same notation but different meaning from the inverse function $f^{-1}: Y \rightarrow X$, which we haven't yet defined, and will define only for some f). In short;

From each map $X \xrightarrow{f} Y$, we get two more maps



THMA For each map $f: X \rightarrow Y$, and each $B \subset \mathcal{S}(Y)$, and $B \subset Y$

$$(1) \quad f^{-1} \cup B = \cup f^{-1}B;$$

$$(2) \quad f^{-1} \cap B = \cap f^{-1}B;$$

if $X_1 = X_2$ & $Y_1 = Y_2$ and $f_1(x) = f_2(x)$ for all $x \in X$.

$$(3) \quad f^{-1}B' = (f^{-1}B)'$$

Proof. (1) $x \in f^{-1}(\bigcup_{B \in \mathcal{B}} B) \Leftrightarrow f(x) \in \bigcup_{B \in \mathcal{B}} B$

$$\Leftrightarrow f(x) \in B, \text{ for some } B \in \mathcal{B}$$

$$\Leftrightarrow x \in f^{-1}B, \quad "$$

$$\Leftrightarrow x \in \bigcup_{B \in \mathcal{B}} f^{-1}B.$$

$$(3) \quad x \in f^{-1}B' \Leftrightarrow f(x) \in B'$$

$$\Leftrightarrow f(x) \notin B$$

$$\Leftrightarrow x \notin f^{-1}B$$

$$\Leftrightarrow x \in (f^{-1}B)'$$

EXERCISE 1 Prove (2) in two ways: (a) similarly to the proof of (1);
(b) by using (1) and (3) and DeMorgan.

THM B For each map $f: X \rightarrow Y$, and each $\mathcal{A} \subset \mathcal{S}(X)$,

$$(1) \quad f(\bigcup_{A \in \mathcal{A}} A) = \bigcup_{A \in \mathcal{A}} fA;$$

$$\frac{1}{2} (2) \quad f(\bigcap_{A \in \mathcal{A}} A) \subset \bigcap_{A \in \mathcal{A}} fA.$$

Proof. (1) $y \in f(\bigcup_{A \in \mathcal{A}} A) \Leftrightarrow y = f(x), \text{ for some } x \in \bigcup_{A \in \mathcal{A}} A$

$$\Leftrightarrow y = f(x), \text{ for some } x \in A, \text{ for some } A \in \mathcal{A}$$

$$\Leftrightarrow y \in fA, \text{ for some } A \in \mathcal{A}$$

$$\Leftrightarrow y \in \bigcup_{A \in \mathcal{A}} fA.$$

$$\frac{1}{2}(2) \quad y \in f\left(\bigcap_{A \in \mathcal{A}} A\right) \Leftrightarrow y = f(x), \text{ for some } x \in \bigcap_{A \in \mathcal{A}} A$$

$$\Rightarrow y \in fA, \text{ for each } A \in \mathcal{A}$$

$$\Leftrightarrow y \in \bigcup_{A \in \mathcal{A}} fA$$

COUNTEREXAMPLE: In general (i.e. sometimes),

$$\bigcap_{A \in \mathcal{A}} fA \neq f\left(\bigcap_{A \in \mathcal{A}} A\right).$$

For example, let $X = \{1, 2\}$, $Y = \{0\}$, $A_1 = \{1\}$, $A_2 = \{2\}$, $f(1) = f(2) = 0$.
 Then $fA_1 \cap fA_2 = \{0\}$ while $A_1 \cap A_2 = \emptyset$, so $f(A_1 \cap A_2) = \emptyset \neq \{0\}$.

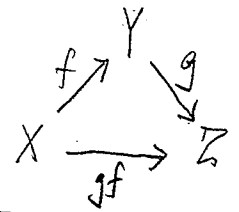
EXERCISE 2. Prove that for each map $f: X \rightarrow Y$,

- (1) $A \subset f^{-1}fA$, for each $A \subset X$;
- (2) $f f^{-1}B \subset B$, for each $B \subset Y$.

Here $f^{-1}fA$ means "the inverse image of the image of A ," etc.

DEF. Given sets X, Y, Z and maps $f: X \rightarrow Y, g: Y \rightarrow Z$, the composite map $gf: X \rightarrow Z$ is defined by

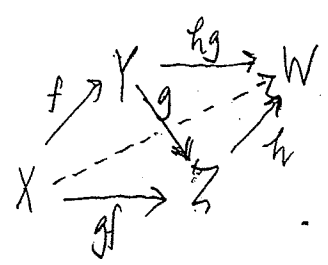
$$(gf)(x) = g(f(x)), \text{ for } x \in X.$$



THM. Composition of maps is associative, i.e.

for all sets X, Y, Z, W and maps $f: X \rightarrow Y, g: Y \rightarrow Z, h: Z \rightarrow W$,

$$h(gf) = (hg)f.$$



Accordingly, we write simply hgf for either of these two equal "double composites," the dotted arrows in the figure.

Proof. For all $x \in X$,

$$(h(gf))(x) = h((gf)(x)) = h(g(f(x))) = (hg)(f(x)) = ((hg)f)(x).$$

COUNTEREXAMPLE. If $f: X \rightarrow Y$ and $g: Y \rightarrow X$, then $gf: X \rightarrow X$ and $fg: Y \rightarrow Y$ are both defined. Even if $X=Y$ these two composites are not necessarily equal. For example, $2(x+1) \neq 2x+1$ shows that for $f, g: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x+1$ and $g(x) = 2x$, $gf \neq fg$.

DEF. Let $f: X \rightarrow Y$ be a map. Then

(1) f is injective if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, i.e. $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

(2) f is surjective if $fX = Y$, i.e. $\forall y \in Y \exists x \in X: f(x) = y$.

(3) f is bijective if f is both injective and surjective.

REMARK. Synonyms are one-to-one, onto, one-to-one and onto.

EXAMPLES. The maps $f, g, h: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = e^x, \quad g(x) = x^3 - x, \quad h(x) = x + 1$$

are (in that order) injective but not surjective, surjective but not injective, bijective. This can be seen, e.g., by drawing the graphs of these functions.

THMD. Given maps $f: X \rightarrow Y$, $g: Y \rightarrow Z$:

(1) f, g injective $\Rightarrow gf$ injective $\Rightarrow f$ injective

(2) f, g surjective $\Rightarrow gf$ surjective $\Rightarrow g$ surjective

(3) f, g bijective $\Rightarrow gf$ bijective.

Proof. (1) Assuming f and g are injective,

$$(gf)(x_1) = (gf)(x_2) \Leftrightarrow g(f(x_1)) = g(f(x_2)) \Rightarrow f(x_1) = f(x_2) \Rightarrow x_1 = x_2, \text{ so } gf \text{ is injective.}$$

\uparrow since g is injective \uparrow since f is injective

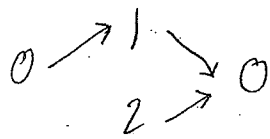
Assuming only gf is injective, if f were not injective then $f(x_1) = f(x_2)$ for some $x_1 \neq x_2$, but then $(gf)(x_1) = (gf)(x_2)$ implying $x_1 = x_2$, a contradiction.

(2) Assuming f and g are surjective, for each $z \in Z$ there is a $y \in Y$ with $g(y) = z$ and there is an $x \in X$ with $f(x) = y$, so $(gf)(x) = g(f(x)) = g(y) = z$, showing that gf is surjective. Assuming only gf is surjective, if g were not surjective there would be $z \in Z$ with $z \neq g(y)$ for any y , but then $z \neq g(f(x)) = (gf)(x)$ for any x , i.e. gf is not surjective, a contradiction.

(3) f, g bijective $\Rightarrow f, g$ injective $\Rightarrow gf$ injective
 $\Rightarrow f, g$ surjective $\Rightarrow gf$ surjective } $\Leftrightarrow gf$ bijective

COUNTEREXAMPLE: gf bijective $\not\Rightarrow f$ surjective
 $\not\Rightarrow g$ injective. In fact

let $X = \{0\}$, $Y = \{1, 2\}$, $Z = \{0\}$, and define $f: X \rightarrow Y$ by $f(0) = 1$,



and $g: Y \rightarrow Z$ by $g(1) = g(2) = 0$. Then gf is bijective, f not surjective, g not injective.

THEM Let $f: X \rightarrow Y$ be a map. Then (with $A' = C_X(A)$, $(fA)' = C_Y(fA)$)

(1) f injective $\Leftrightarrow fA' \subset (fA)'$ for each $A \subset X$

(2) f surjective $\Leftrightarrow fA' \supset (fA)'$ for each $A \subset X$

(3) f bijective $\Leftrightarrow fA' = (fA)'$ for each $A \subset X$

EXERCISE 3. Prove each of the six assertions. Hints: f injective $\Rightarrow fA \cap (fA)' = \emptyset$, $\forall A \subset X$; f surjective $\Rightarrow fA \cup (fA)' = Y$, $\forall A \subset X$. For (1) \Leftarrow use $A = \{x\}$, $\forall x \in X$. For (2) \Leftarrow use $A = \emptyset$.