4. Algebra of Maps

DEF. Given sets X and Y, a <u>function</u> i.e. <u>map</u> $f : X \to Y$ is any rule which associates to each $x \in X$ some unique $y \in Y$ which (usually) depends on x, and is denoted by $f(x)$.

if $X_1 = X_2$ & $Y_1 = Y_2$ and $f_1(x) = f_2(x)$ for all $x \in X_1$.

EXAMPLE For $X = \mathbb{R}$ and $Y = \mathbb{R}$, $f(x) = x^2$ defines a map $f : \mathbb{R} \to \mathbb{R}$.
For any X and Y and $c \in Y$, $f(x) = c$ for all $x \in X$ defines a <u>constant map</u>.

DEF Given sets X and Y, and a map $f : X \to Y$,

(1) For each $A \subset X$, the <u>image</u> fA of A by f is the subset of Y defined by

$$y \in fA \iff y = f(x) \text{ for some } x \in A;$$

(2) for each $B \subset Y$, the <u>inverse image</u> $f^{-1}B$ of B by f is the subset of X defined by

$$x \in f^{-1}B \iff f(x) \in B.$$

REMARK Recall the notation $\mathcal{S}(X)$ for the set of all subsets of X.
In the above definition we have defined, for each map $f : X \to Y$
two new maps, one from $\mathcal{S}(X)$ to $\mathcal{S}(Y)$, taking $A \subset X$ to $fA \subset Y$,
the other from $\mathcal{S}(Y)$ to $\mathcal{S}(X)$, taking $B \subset Y$ to $f^{-1}B \subset X$.
We will denote the first of these by f (same notation but different meaning from the $f : X \to Y$ we started with), and the second by f^{-1} (same notation but different meaning from the inverse function $f^{-1} : Y \to X$, which we haven't yet defined, and will define only for some f). In short:

From each map $X \xrightarrow{f} Y$, we get two more maps
$$\begin{tikzcd} \mathcal{S}(X) \arrow[r, bend left, "f"] & \mathcal{S}(Y) \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

THM A For each map $f : X \to Y$, and each $\mathcal{B} \subset \mathcal{S}(Y)$, and $B \subset Y$

(1) $$f^{-1} \bigcup_{B \in \mathcal{B}} B = \bigcup_{B \in \mathcal{B}} f^{-1} B;$$

(2) $$f^{-1} \bigcap_{B \in \mathcal{B}} B = \bigcap_{B \in \mathcal{B}} f^{-1} B;$$

(3) $\quad f^{-1}B' = (f^{-1}B)'$.

Proof. (1) $\quad x \in f^{-1}\left(\bigcup_{B\in\mathcal{B}} B\right) \Leftrightarrow f(x) \in \bigcup_{B\in\mathcal{B}} B$

$$\Leftrightarrow f(x) \in B, \text{ for some } B \in \mathcal{B}$$

$$\Leftrightarrow x \in f^{-1}B, \quad \text{"}$$

$$\Leftrightarrow x \in \bigcup_{B\in\mathcal{B}} f^{-1}B.$$

(3) $\quad x \in f^{-1}B' \Leftrightarrow f(x) \in B'$

$$\Leftrightarrow f(x) \notin B$$

$$\Leftrightarrow x \notin f^{-1}B$$

$$\Leftrightarrow x \in (f^{-1}B)'.$$

EXERCISE 1 Prove (2) in two ways: (a) similarly to the proof of (1); (b) by using (1) and (3) and DeMorgan.

THM B For each map $f: X \to Y$, and each $\mathcal{A} \subset \mathcal{S}(X)$,

(1) $\quad f\left(\bigcup_{A\in\mathcal{A}} A\right) = \bigcup_{A\in\mathcal{A}} fA$;

$\frac{1}{2}$ (2) $\quad f\left(\bigcap_{A\in\mathcal{A}} A\right) \subset \bigcap_{A\in\mathcal{A}} fA$.

Proof. (1) $\quad y \in f\left(\bigcup_{A\in\mathcal{A}} A\right) \Leftrightarrow y = f(x), \text{ for some } x \in \bigcup_{A\in\mathcal{A}} A$

$$\Leftrightarrow y = f(x), \text{ for some } x \in A, \text{ for some } A \in \mathcal{A}$$

$$\Leftrightarrow y \in fA, \text{ for some } A \in \mathcal{A}$$

$$\Leftrightarrow y \in \bigcup_{A\in\mathcal{A}} fA.$$

$\frac{1}{2}(2)\quad y \in f\left(\bigcap_{A\in\mathcal{A}} A\right) \Leftrightarrow y = f(x)$, for some $x \in \bigcap_{A\in\mathcal{A}} A$

$\Rightarrow y \in fA$, for each $A \in \mathcal{A}$

$\Leftrightarrow y \in \bigcup_{A\in\mathcal{A}} fA$

COUNTEREXAMPLE: In general (i.e. sometimes),

$$\bigcap_{A\in\mathcal{A}} fA \not\subset f \bigcap_{A\in\mathcal{A}} A.$$

For example, let $X=\{1,2\}$, $Y=\{0\}$, $A_1=\{1\}$, $A_2=\{2\}$, $f(1)=f(2)=0$.
Then $fA_1 \cap fA_2 = \{0\}$ while $A_1 \cap A_2 = \phi$, so $f(A_1\cap A_2) = \phi \not\ni 0$.

EXERCISE 2. Prove that for each map $f: X \to Y$,

(1) $A \subset f^{-1}fA$, for each $A \subset X$;

(2) $ff^{-1}B \subset B$, for each $B \subset Y$.

Here $f^{-1}fA$ means "the inverse image of the image of A," etc.

DEF Given sets X, Y, Z and maps $f: X\to Y$, $g: Y \to Z$, the composite map $gf: X \to Z$ is defined by

$$(gf)(x) = g(f(x)), \text{ for } x \in X.$$

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "gf"'] & & Z \end{tikzcd}$$

THM Composition of maps is associative, i.e. for all sets X, Y, Z, W and maps $f: X\to Y$, $g: Y \to Z$, $h: Z \to W$,

$$h(gf) = (hg)f.$$

Accordingly, we write simply hgf for either of these two equal "double composites," the dotted arrows in the figure.

$$\begin{tikzcd} & Y \arrow[r, "hg"] \arrow[d, "g"] & W \\ X \arrow[ur, "f"] \arrow[r, "gf"'] \arrow[urr, dashed] & Z \arrow[ur, "h"'] & \end{tikzcd}$$

Proof. For all $x \in X$,

$$(h(gf))(x) = h((gf)(x)) = h(g(f(x))) = (hg)(f(x)) = ((hg)f)(x).$$

COUNTEREXAMPLE. If $f: X \to Y$ and $g: Y \to X$, then $gf: X \to X$ and $fg: Y \to Y$ are both defined. Even if $X = Y$ these two composites are not necessarily equal. For example, $2(x+1) \neq 2x+1$ shows that for $f, g: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x+1$ and $g(x) = 2x$, $gf \neq fg$.

DEF. Let $f: X \to Y$ be a map. Then

(1) f is <u>injective</u> if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, i.e. $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

(2) f is <u>surjective</u> if $fX = Y$, i.e. $\forall y \in Y \; \exists x \in X : f(x) = y$.

(3) f is <u>bijective</u> if f is both injective and surjective.

REMARK. Synonyms are <u>one-to-one</u>, <u>onto</u>, <u>one-to-one and onto</u>.

EXAMPLES. The maps $f, g, h: \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = e^x, \quad g(x) = x^3 - x, \quad h(x) = x+1$$

are (in that order) injective but not surjective, surjective but not injective, bijective. This can be seen, e.g., by drawing the graphs of these functions.

THMD. Given maps $f: X \to Y$, $g: Y \to Z$:

(1) f, g injective $\Rightarrow gf$ injective $\Rightarrow f$ injective

(2) f, g surjective $\Rightarrow gf$ surjective $\Rightarrow g$ surjective

(3) f, g bijective $\Rightarrow gf$ bijective.

Proof. (1) Assuming f and g are injective,

$$(gf)(x_1)=(gf)(x_2) \Leftrightarrow g(f(x_1))=g(f(x_2)) \underset{\substack{\uparrow \\ \text{since } g \text{ is injective}}}{\Rightarrow} f(x_1)=f(x_2) \underset{\substack{\uparrow \\ \text{since } f \text{ is injective}}}{\Rightarrow} x_1=x_2,$$ so gf is injective.

Assuming only gf is injective, if f were not injective then $f(x_1)=f(x_2)$ for some $x_1 \neq x_2$, but then $(gf)(x_1)=(gf)(x_2)$ implying $x_1=x_2$, a contradiction.

(2) Assuming f and g are surjective, for each $z \in Z$ there is a $y \in Y$ with $g(y)=z$, and there is an $x \in X$ with $f(x)=y$, so $(gf)(x)=g(f(x))=g(y)=z$, showing that gf is surjective. Assuming only gf is surjective, if g were not surjective there would be $z \in Z$ with $z \neq g(y)$ for any y, but then $z \neq g(f(x))=(gf)(x)$ for any x, i.e. gf is not surjective, a contradiction.

(3) $$f,g \text{ bijective} \begin{array}{l} \nearrow f,g \text{ injective} \Rightarrow gf \text{ injective} \\ \searrow f,g \text{ surjective} \Rightarrow gf \text{ surjective} \end{array} \Bigg\} \Leftrightarrow gf \text{ bijective}$$

COUNTEREXAMPLE. gf bijective $\begin{array}{l} \not\Rightarrow f \text{ surjective} \\ \not\Rightarrow g \text{ injective} \end{array}$. In fact let $X=\{0\}$, $Y=\{1,2\}$, $Z=\{0\}$, and define $f: X \to Y$ by $f(0)=1$,

$$\begin{array}{ccccc} & & 1 & & \\ 0 & \nearrow & & \searrow & 0 \\ & & 2 & \nearrow & \end{array}$$

and $g: Y \to Z$ by $g(1)=g(2)=0$. Then gf is bijective, f not surjective, g not injective.

THM E Let $f: X \to Y$ be a map. Then (with $A'=C_X(A)$, $(fA)'=C_Y(fA)$)

(1) f injective $\Leftrightarrow fA' \subset (fA)'$ for each $A \subset X$

(2) f surjective $\Leftrightarrow fA' \supset (fA)'$ for each $A \subset X$

(3) f bijective $\Leftrightarrow fA' = (fA)'$ for each $A \subset X$

EXERCISE 3. Prove each of the six assertions. Hints: f injective $\Rightarrow fA \cap fA' = \phi$, $\forall A \subset X$; f surjective $\Rightarrow fA \cup fA' = Y$, $\forall A \subset X$. For (1) $\Leftarrow$ use $A=\{x\}$, $\forall x \in X$. For (2) $\Leftarrow$ use $A=\phi$.