§4 Algebra of Sets

NOTATION Usually, capital letters denote sets and small letters elements.

$x \in A$ means "x is an element of A" i.e. "x is in A."

$x \notin A$ means "x is not an element of A"

$A = \{x_1, x_2, \ldots\}^{*)}$ means "A consists of the elements $x_1, x_2, \ldots$."

$A = \{x : x \text{ has property } \mathcal{P}\}$ means A consists of all x having property $\mathcal{P}$.

Observe, without too much enthusiasm, that $A = \{x : x \in A\}$.

EXAMPLES of Sets: $\mathbb{N}, \mathbb{Z}; \mathbb{Q}, \mathbb{R}, \mathbb{C}$

$\mathbb{N}$ and $\mathbb{Z}$ were defined on 1.1

$\mathbb{Q}$ is the set of rational numbers, i.e. fractions, i.e.

$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z} \text{ and } n \in \mathbb{N}\}$.

$\mathbb{R}$ (and $\mathbb{C}$) are the sets of all real (and complex) numbers

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DEF For sets A and B,

(1). $A = B$ means "A and B consist of the same elements", i.e. $x \in A \Leftrightarrow x \in B$.

(2) $A \subset B$ means "Each element of A is an element of B", i.e. $x \in A \Rightarrow x \in B$

(3) $A \supset B$ means $B \subset A$, i.e. $x \in B \Rightarrow x \in A$, i.e. $x \in A \Leftarrow x \in B$

THM A $\quad A = B \Leftrightarrow A \subset B$ and $A \supset B$.

Proof. This follows from the three circled conditions, since, for any two statements $\mathcal{a}, \mathcal{B}$

$$\mathcal{a} \Leftrightarrow \mathcal{B} \text{ means } \mathcal{a} \Rightarrow \mathcal{B} \text{ and } \mathcal{a} \Leftarrow \mathcal{B}.$$

The statement $A \subset B$ is read "A is a <u>subset</u> of B."

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*) Although sometimes the elements of a set may be listed in some order, order doesn't count; e.g. $\{x, y\} = \{y, x\}$.

THM B. For any three sets A, B, C, if $A \subset B$ and $B \subset C$, then $A \subset C$.

Proof This follows from (2.) in the definition on 3.1, since

$$\mathcal{A} \Rightarrow \mathcal{B} \ \& \ \mathcal{B} \Rightarrow \mathcal{C} \Rightarrow \mathcal{A} \Rightarrow \mathcal{C},$$

for any three statements $\mathcal{A}, \mathcal{B}, \mathcal{C}$.

DEF. $A \subsetneqq B$ means $A \subset B$ but $A \neq B$.

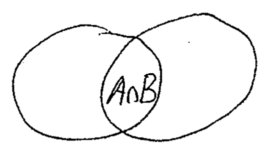
EXAMPLES $\mathbb{N} \subsetneqq \mathbb{Z} \subsetneqq \mathbb{Q} \subsetneqq \mathbb{R} \subsetneqq \mathbb{C}$.

To see the $\neq$'s here, observe that $0 \in \mathbb{Z}$ but $0 \notin \mathbb{N}$, $\frac{1}{2} \in \mathbb{Q}$ but $\frac{1}{2} \notin \mathbb{Z}$, $\sqrt{2}$ is real but irrational *), and the complex number i, which satisfies $i^2 = -1$, is not real

DEF. For any two sets A & B, their union and intersection are

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$



THM C. For any three sets A, B, C,

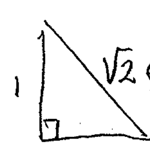
(1) $A \cup B = B \cup A$ & $A \cap B = B \cap A$

(2) $A \cup (B \cup C) = (A \cup B) \cup C$ & $A \cap (B \cap C) = (A \cap B) \cap C$

(3) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ & $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,

i.e. $\cup$ and $\cap$ are commutative, associative and very distributive, more so than $+$ & $\cdot$ with numbers: While $a(b+c) = ab + ac$, $a + bc \neq (a+b)(a+c)$ generally.

*) This ≃2000 year old discovery spooked the Pythagoreans:



$\sqrt{2} \notin \mathbb{Q}$, whoa!

If $\sqrt{2}$ were rational, it could be written as a reduced fraction.
In particular, $\sqrt{2} = \frac{m}{n}$ with $m, n \in \mathbb{N}$, not both even. But then $2n^2 = m^2$, so m is even, $m = 2m_1$, $n^2 = 2m_1^2$, so n is even, a contradiction.

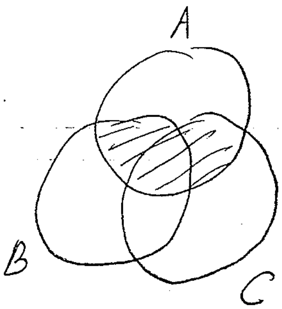
Proof of left equation in (3):

$x \in A \cap (B \cup C) \Leftrightarrow x \in A$ and ($x \in B$ or $x \in C$)

$\Leftrightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

$\Leftrightarrow$ ($x \in A \cap B$) or ($x \in A \cap C$)

$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$.



Therefore, by the definition on 3.1(1) of equality of sets,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

DEF For any collection of sets A_i ($i \in I$, an "index set")

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}$$

$$\bigcap_{i \in I} A_i = \{x : x \in A_i \text{ for all } i \in I\}.$$

THM D. For A_i as above, and any set B,

$$* \quad \Big(\bigcup_{i \in I} A_i\Big) \cap B = \bigcup_{i \in I} (A_i \cap B)$$

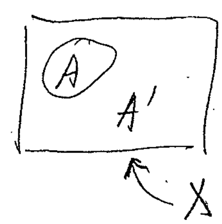
$$** \quad \Big(\bigcap_{i \in I} A_i\Big) \cup B = \bigcap_{i \in I} (A_i \cup B)$$

EXERCISE 1. Prove * and **

DEF. For a while, we consider only subsets $A \subset X$ with X fixed, i.e. some given X.

The complement of A (relative to X), denoted $C_X(A)$ or CA or A' is

$$\boxed{A' = \{x \in X : x \notin A\}}$$



THM E (De Morgan's Rules) For $A, B \subset X$,

(1) $(A \cup B)' = A' \cap B'$

(2) $(A \cap B)' = A' \cup B'$

(3) $(A')' = A$.

Proof (1). $x \in (A \cup B)' \Leftrightarrow x \notin A \cup B$

$$\Leftrightarrow x \notin A \ \& \ x \notin B$$

$$\Leftrightarrow x \in A' \ \& \ x \in B'$$

$$\Leftrightarrow x \in A' \cap B'.$$

Therefore $(A \cup B)' = A' \cap B'$.

EXERCISE 2. Prove (2) similarly.

Proof (3). $x \in (A')' \Leftrightarrow x \notin A'$ i.e. x is not not in $A \Leftrightarrow x \in A$.

Another proof of (2), which uses rules (1) and (3):

$$A \cap B = A'' \cap B'' = (A' \cup B')'$$

(by (1) applied to A' & B').

Therefore

$$(A \cap B)' = (A' \cup B')'' = A' \cup B'.$$

EXERCISE 3 Prove $\left(\bigcup_{i \in I} A_i\right)' = \bigcap_{i \in I} A_i'$ and $\left(\bigcap_{i \in I} A_i\right)' = \bigcup_{i \in I} A_i'$

DEF. The empty set, denoted by ϕ, has no elements.

It is a subset of every set: In fact, for each set X, $\phi \subset X$, because

if $x \in \phi$ then $x \in A$, since there are no $x \in \phi$, so there is nothing to show.

THM F. For each set $A \subset X$,

$$X \cup A = X, \quad \phi \cup A = A;$$

$$X \cap A = A, \quad \phi \cap A = \phi;$$

Also $\quad X' = \phi, \quad \phi' = X.$

DEF Sets A and B are disjoint if $A \cap B = \phi$. A B

In this case we will sometimes write $A \cup B = A + B$ (the disjoint union).

NOTATION If A is a finite set, $\#A$ denotes the number of elements in A.

NOTATION. For any set A, $\mathcal{S}(A)$ denotes the set of all subsets of A.

EXAMPLE For $A=\{1,2\}$, $\mathcal{S}(A)=\{\phi, \{1\}, \{2\}, \{1,2\}\}$.

How many elements does ϕ have? None, so $\#\phi = 0$.
How many subsets does ϕ have? Only one: ϕ itself, so $\#\mathcal{S}(\phi)=1$.

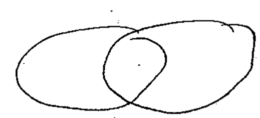
THM G If $\#A=n$, then $\#\mathcal{S}(A)=2^n$.

Proof. We may suppose $A=A_n=\{1,2,\dots,n\}$, so $A=A_{n-1}+\{n\}$. For $n\geq 1$, each subset $B\subset A_n$ either contains $\{n\}$ or not, and the rest of B is a subset of A_{n-1}. It follows that $\#\mathcal{S}A_n = 2\#\mathcal{S}A_{n-1}$, so by induction, $\#\mathcal{S}A_n = 2\cdot 2^{n-1} = 2^n$. Of course we must check that $\#\mathcal{S}A_1 = 2$ ($\mathcal{S}A_1 = \{\phi, \{1\}\}$).

REMARK. THM G also holds for $n=0$, since $2^0=1$ and $\mathcal{S}(\phi)=\{\phi\}$.

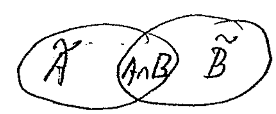
THM H. If A and B are finite sets, then

$$\boxed{\#(A\cup B)+\#(A\cap B)=\#A+\#B}$$



Proof. As shown at right,

$$A=\tilde{A}+(A\cap B)\ \&\ B=\tilde{B}+(A\cap B),$$



so

✓ $\quad \#A=\#\tilde{A}+\#(A\cap B)\ \&\ \#B=\#\tilde{B}+\#(A\cap B)$

Also

$$A\cup B=\tilde{A}+A\cap B+\tilde{B},$$

so

✓✓ $\quad \#(A\cup B)=\#\tilde{A}+\#(A\cap B)+\#\tilde{B}$

Hint: $A\cup B\cup C$ " $A\cup(B\cup C)$, etc, etc

Combining ✓ & ✓✓ we get $\square$.

EXERCISE 4. Use H to prove $\#(A\cup B\cup C)=\#A+\#B+\#C-\#(A\cap B)-\#(A\cap C)-\#(B\cap C)+\#(A\cap B\cap C)$