§3 Chebychev's Estimates for $\pi(x)$

Half way between Gauss' conjecture and the proof of the Prime Number Theorem, Chebychev was able to prove a weaker version of PNT. This was one of his biggest achievements, and, for 50 years, the most that was known about $\pi(x)$.

THM (Chebychev, 1850) There are positive constants c & C so that

$$\boxed{c\frac{x}{\log x} \leq \pi(x) \leq C\frac{x}{\log x}, \text{ for } x \geq 2.}$$

The proof that $\pi(x) \to \infty$ took two lines. This takes four pages.

LEMMA (About the middle binomial coefficient). For all $N \in \mathbb{N}$,

(a) $$\frac{2^{2N}}{2N} \leq \binom{2N}{N} \leq 2^{2N};$$

(b) $$N^{\pi(2N)-\pi(N)} \leq \binom{2N}{N} \leq (2N)^{\pi(2N)}.$$

Proof (a). According to the Binomial Theorem, for $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \quad \text{with } \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Putting $n = 2N$ and $x = 1$, this gives

$$\checkmark \qquad 2^{2N} = \sum_{k=0}^{2N} \binom{2N}{k},$$

from which the right inequality in (a) follows.

Observe that

$$\binom{n}{k-1} < \binom{n}{k} = \binom{n}{n-k} \quad \text{for } 0 < k \leq \frac{n}{2}.$$

It follows that the middle term in ✓ is largest and is ≥ 2 (= sum of terms with $k=0$ & $k=2N$). Thus the sum in ✓ is $\leq 2N\binom{2N}{N}$, proving the left inequality in (a).

EXERCISE 1. Prove the observation $<$, using the formulas for $\binom{n}{k}$ and $\binom{n}{k-1}$.

(b) Since $(N!)^2\binom{2N}{N}=(2N)!$, and $n!$ is divisible by each prime $p\le n$ and no prime $p>n$, it follows that $\binom{2N}{N}$ is divisible by each prime p with $N<p\le 2N$, and therefore by their product, so

$$\binom{2N}{N}\ge\prod_{N<p\le 2N} p \ge N^{\pi(2N)-\pi(N)},$$

giving the left inequality in (b).

To prove the right inequality in (b) we first get the prime factorization of $n!$:

$$\checkmark \qquad n! = \prod p^{\sigma_p(n)}, \quad \text{with } \sigma_p(n)=\sum_{r\in\mathbb{N}}\left[\frac{n}{p^r}\right].$$

To see this, observe first that the multiplicity of a prime p in the prime factorization of a positive integer k equals the number of positive integers r with $p^r|k$.
It follows that the multiplicity of p in $n!$ equals the number of pairs of positive integers r, k for which $p^r|k$ and $k\le n$. For fixed r, there are $[\frac{n}{p^r}]$ positive integers k with $p^r|k$ and $k\le n$.

It follows from $\checkmark$ that

$$\checkmark\checkmark \qquad \binom{2N}{N}=\prod p^{\tau_p(n)}, \quad \text{with } \tau_p(n)=\sum_{r\in\mathbb{N}}\left(\left[\frac{2N}{p^r}\right]-2\left[\frac{N}{p^r}\right]\right).$$

Observe next that the function θ defined on $\mathbb{R}$ by

$$\theta(u)=[2u]-2[u]$$

satisfies $\theta(u+1)=\theta(u)$ (since $[v+1]=[v]+1$), and $\theta(u)=0$ on $[0,\frac{1}{2})$, 1 on $[\frac{1}{2},1)$, so $0\le\theta(u)\le 1$ for all $u\in\mathbb{R}$. Since only terms with $p^r\le 2N$ can contribute to $\tau_p(n)$, we get $\tau_p(n)\le\left[\frac{\log 2N}{\log p}\right]$. Therefore, from $\checkmark\checkmark$,

$$\binom{2N}{N}\le\prod_{p\le 2N} p^{\frac{\log 2N}{\log p}} = (2N)^{\pi(2N)},$$

giving the right inequality in (b).

EXERCISE 2. Prove the (=).

Proof of Chebyshev's Theorem: The four inequalities in the lemma can be combined, in two different ways, to give information (* & ** below) about $\pi(x)$:

Combining the left inequality in (a) with the right inequality in (b) gives

$$\frac{2^{2N}}{2N} \leq (2N)^{\pi(2N)}.$$

Taking logs (& using $\log u \leq \log v$ for $u \leq v$), we get

$$\pi(2N) \geq (\log 2)\frac{2N}{\log 2N} - 1, \quad (50)$$

$$* \quad \boxed{\pi(2N) \geq b\,\frac{2N}{\log 2N}, \text{ for all } N \in \mathbb{N}},$$

with $b = \frac{\log 2}{2}$, since $\frac{\log 2}{2} (\geq) \frac{\log 2N}{2N}$ for $N \in \mathbb{N}$.

EXERCISE 3. Verify (≥) by showing that $\frac{\log x}{x}$ decreases for $x \geq e$ and $\frac{\log 2}{2} = \frac{\log 4}{4}$. Then show how to use ↑ to derive * from the inequality just above *, just before (50).

Combining the left inequality in (b) with the right inequality in (a) gives

$$N^{\pi(2N)-\pi(N)} \leq 2^{2N}.$$

Taking logs,

$$\pi(2N) - \pi(N) \leq (\log 4)\frac{N}{\log N}.$$

We will use this to show that there is a positive constant B so that

$$** \quad \boxed{\pi(2^k) \leq B\,\frac{2^k}{\log 2^k}, \text{ for all } k \in \mathbb{N}}.$$

For $k = 1$ or 2 this is easy, with $B = \log 2$. We try induction. For $k \geq 2$, the inequality just above ** with $N = 2^k$ gives

$$\pi(2^{k+1}) = \pi(2^{k+1}) - \pi(2^k) + \pi(2^k)$$

$\log 2^k$ means $\log(2^k)$

$$\leq (\log 4)\frac{2^k}{\log 2^k} + B\frac{2^k}{\log 2^k} \overset{?}{\leq} B\frac{2^{k+1}}{\log 2^{k+1}}$$

Here we have supposed that $**$ is true for k, with some B, and ask if it is true for $k+1$, with the same B. Cancelling $\frac{2^k}{\log 2}$ from the questioned last inequality gives

$$\frac{\log 4}{k} + \frac{B}{k} \overset{?}{\leq} \frac{2B}{k+1},$$

Multiplying by $k(k+1)$ gives

$$(\log 4)(k+1) \overset{?}{\leq} (k-1)B,$$

i.e.

$$B \overset{?}{\geq} (\log 4)\frac{k+1}{k-1}.$$

For $k \geq 2$ the right side is largest for $k=2$, so, yes, it works, for $B = 3\log 4$.

Finally, we use $*$ and $**$ to finish the proof of Chebyshev's Theorem.

For $x \geq 2$, choose $N \in \mathbb{N}$ with $2N \leq x \leq 4N$. Then by $(*)$,

$$\pi(x) \geq \pi(2N) \geq b\frac{2N}{\log 2N} = \frac{b}{2}\frac{4N}{\log 2N} \geq c\frac{x}{\log x},$$

with $c = \frac{b}{2}\left(= \frac{\log 2}{4} > 0.17\right)$, giving the left inequality in the theorem.

For $x \geq 2$, choose $k \in \mathbb{N}$ with $2^k \leq x \leq 2^{k+1}$. Then by $**$,

$$\pi(x) \leq \pi(2^{k+1}) \leq B\frac{2^{k+1}}{\log 2^{k+1}} = 2B\frac{2^k}{\log 2^{k+1}} \leq C\frac{x}{\log x},$$

with $C = 2B\ (= 12\log 2 < 8.4)$.

For more on $\pi(x)$, see H. Davenport's <u>Multiplicative Number Theory</u>.