§25 Appendix Euler's Polyhedron Theorem, with Legendre's Proof

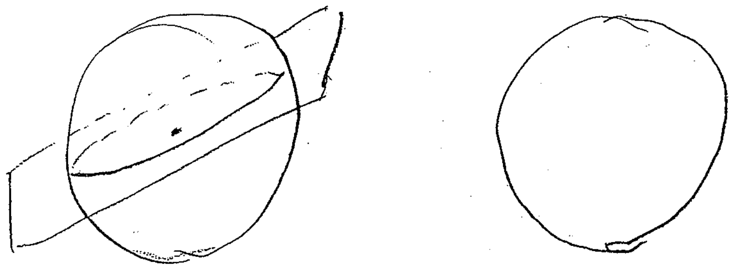
THM A (Stated by Euler, proved by Legendre, 1700's).
Let F, E, V be the number of faces, vertices, edges in a convex*) polyhedron. Then

$$\boxed{F - E + V = 2}$$

EXAMPLES Tetrahedron: $4-6+4$
Cube: $6-12+8$ Octahedron $8-12+6$
Dodecahedron: $12-30+20$ Icosahedron $20-30+12$

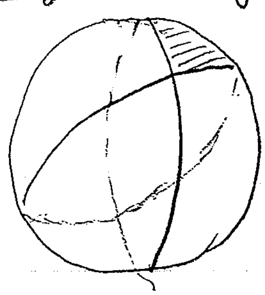
LEMMA 1 (Archimedes) The surface area of a sphere of radius 1 is 4π.

DEF. A *great circle* on the surface of a sphere is the intersection of the spherical surface with a plane through the center of the sphere



great circle from a convenient point of view

A *spherical triangle* is made of three arcs of great circles

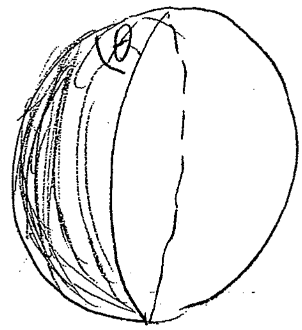


one great circle drawn conveniently.

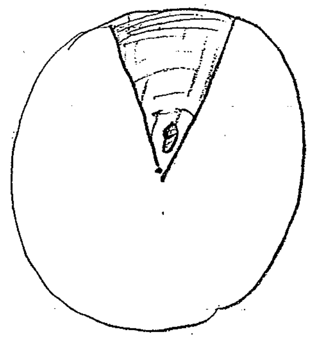
THM (Girard, 1629) The area of a spherical triangle on the surface of a sphere of radius 1 is equal to $\alpha+\beta+\gamma-\pi$, where α, β, γ are the angles of the triangle at its vertices.

*) look up *convex*. What we will use is that a point light source inside bijects the surface of the polyhedron to the surface of any containing sphere.

Proof. First lets calculate the area of a lune, the region between two spherical semicircles



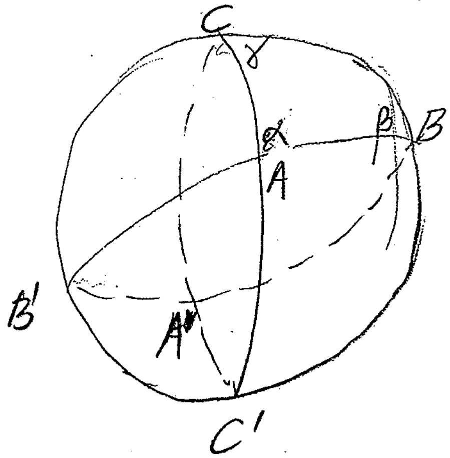
one circle drawn conveniently



another view showing half of the lune.

$$\left[\text{area} = \frac{\theta}{2\pi}\cdot 4\pi = \boxed{2\theta.}\right].$$

Now lets look at the 8 spherical triangles below,



four you can see on the hemisphere in front, and four you can't see, on the other hemisphere.
As areas,

$$\left.\begin{aligned} ABC &= A'B'C' \\ ABC' &= A'B'C \\ AB'C &= A'BC' \\ A'BC &= AB'C', \end{aligned}\right] \text{ and } \quad \begin{aligned} 2\alpha &= ABC + A'BC \\ 2\beta &= ABC + AB'C \\ 2\gamma &= ABC + ABC' \end{aligned}$$

(by area $= 2\theta$)

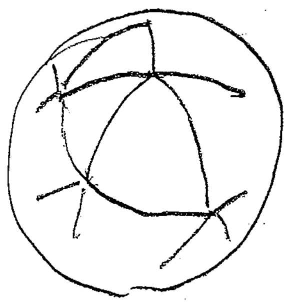
Therefore

$$2(\alpha+\beta+\gamma) = 2ABC + \underbrace{(ABC + \underbrace{A'BC}_{AB'C'} + AB'C + ABC')}_{=\frac{1}{2}4\pi}$$

the last step because the last four terms give the area of a hemisphere

REMARK Any line segment inside the sphere is part of a plane through the center, so a light source at the center will project the line segment to an arc of a great circle on the sphere. It follows that any triangle inside the sphere projects from the center of the sphere to a spherical triangle.

Legendre's Proof of Euler's Theorem: Put the polyhedron in a unit sphere so that the center of the sphere is inside the polyhedron. Suppose the surface of the sphere is translucent. Turn out all the lights in the room, and draw the curtains if it is daytime. In the darkened room, turn on a point light source at the center of the sphere. The edges of the polyhedron will cast shadows on the glowing spherical surface which are the edges of a spherical polyhedron made of spherical triangles (if the original polyhedron has triangular faces)



canals on Mars

$$4\pi = \sum(\text{spherical triangle areas}) = \sum_{\text{faces}}(\alpha_u + \beta_u + \gamma_u - \pi)$$

Since the angles around each vertex sum to 2π, this gives

$$4\pi = 2\pi V - \pi F,$$

or

$$* \qquad 2 = V - \tfrac{1}{2}F$$

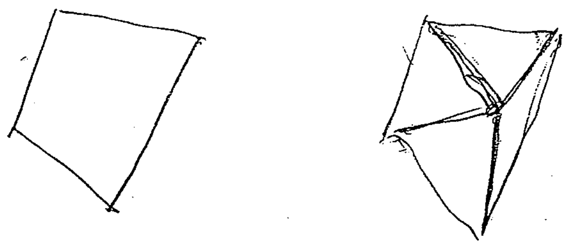
But each face has 3 edges and each edge borders 2 faces, so

$$** \qquad 2E = 3F.$$

Combining * & **,

$$F - E + V = V - \tfrac{1}{2}F = 2$$

Finally, if the faces of the original polyhedron are not triangles, triangulate them:



For a face with e edges this introduces e more edges $e-1$ more faces and 1 new vertex, so doesn't change $F-E+V$. Thus we are reduced to the case of polyhedra with triangular faces

EXERCISE Find F, E, V for all convex polyhedra

all of whose faces are p-gons

and

all of whose vertices are q-valent

for each possible choice of p & q $(\in \{3,4,5,6,\ldots\})$

You will find, using Euler's Theorem that you only get the numbers you have seen before, with

tetrahedron, cube, octahedron, dodecahedron, icosahedron.