In how many ways can the faces of a cube be colored, if there are 3 available colors, there is to be one color per face, and we agree to count two colorings as the same if a rotation stabilizing the cube takes one to the other?

Without the agreement, since there are 6 faces, each of which can be colored in any of 3 ways, there are $3^6 = 729$ colorings.

With the agreement, there should be fewer colorings

Here is a general context,[*)] due to Polya (1937) and Redfield (?) for answering this and analogous questions:

THM·A Let X and Y be finite sets, and let G be a finite group with a given action of G on X. Then G acts on the set $\mathcal{M}(X,Y)$ of all maps $f: X \to Y$ by $\boxed{(af)(x) = f(a^{-1}x)}$ for $a \in G$, $f \in \mathcal{M}(X,Y)$, $x \in X$, and the number of orbits in this action is

$$|G \setminus \mathcal{M}(X,Y)| = \frac{1}{|G|} \sum_{a \in G} |Y|^{\nu(a)},$$

where $\nu(a) (= \nu(\pi(a)))$ denotes the number of cycles in the permutation $x \mapsto ax$ of X.

Proof. We first verify that $\Box$ defines an action of G on $\mathcal{M}(X,Y)$:

For all $f \in \mathcal{M}(X,Y)$ and $x \in X$,

$$(1f)(x) = f(1^{-1}x) = f(1x) = f(x),$$

so $1f = f$.

For all $f \in \mathcal{M}(X,Y)$, $a, b \in G$ and $x \in X$,

*) For lots more good reading on this, google Polya Theory.

$(ab.f)(x) = f((ab)^{-1}x) = f((b^{-1}a^{-1})x) = f(b^{-1}(a^{-1}x)) = (bf)(a^{-1}x) = (a.bf)(x)$,

so $ab.f = a.bf$.

Thus we have an action of G on $\mathcal{M}(X,Y)$

By the Cauchy–Frobenius Theorem,

$$(*) \quad |G\backslash \mathcal{M}(X,Y)| = \frac{1}{|G|}\sum_{a\in G} \theta(a),$$

where $\theta(a)$ is the number of fixed points in this action, ie

$$\theta(a) = |\{f\in \mathcal{M}(X,Y) : af = f\}|$$

The condition $\bigcirc$ is equivalent to

$$f(a^{-1}x) = f(x) \quad \forall x\in X$$

ie. to

$$f \text{ is constant on each cycle of the permutation } x\mapsto ax$$

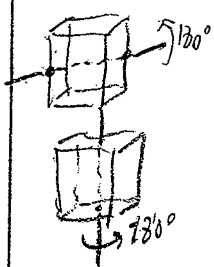
Thus an $f\in \mathcal{M}(X,Y)$ satisfying $\bigcirc$ is really a map from the set of all $\nu(a)$ cycles of this permutation to Y. There are $|Y|^{\nu(a)}$ such maps, so

$$(**) \quad \theta(a) = |Y|^{\nu(a)}.$$

Combining $(*)$ & $(**)$ gives the formula in the Theorem.

EXAMPLES. The group $G = R_C$ of all rotations stabilizing the cube C acts on the set X of all faces of C. If $Y = \{$yellow, orange, red$\}$, then the number of ways of coloring the faces using colors from Y, is, subject to our agreement,

$$\frac{1}{24}\sum_{a\in R_C} 3^{\nu(a)}, \quad \text{where } \nu(a) = \#\text{ of cycles of } a \text{ on the set of faces}$$



order of a	number of such a	cycle type of a on faces	$\nu(a)$
1	1	1+1+1+1+1+1	6
2_{ee}	6	2+2+2	3
2_{ff}	3	2+2+1+1	4
3	8	3+3	2
4	6	4+1+1	3
	24 ✓		

Thus the number of colorings of faces of the cube with 3 available colorings, with our agreement, is

$$\frac{1}{24}\left(1.3^6 + 6.3^3 + 3.3^4 + 8.3^2 + 6.3^3\right)$$

$$729 \quad 162 \quad 243 \quad 72 \quad 162$$

$$\begin{array}{r} 729 \\ 162 \\ 243 \\ 72 \\ 162 \\ \hline 1368 \end{array}$$

$$\begin{array}{r} \boxed{57} \\ 24\,\overline{)1368} \\ 120 \\ \hline 168 \\ 168 \\ \hline 0 \end{array} \; !$$

It is remarkable! that a fraction with denominator 24 actually reduces to an integer, but that's what must happen here since the number we are calculating here counts the number of elements in a finite set

It is no harder to answer the corresponding question for coloring (with any given number of colors) faces, or edges, or vertices of a cube, or regular tetrahedron, or regular dodecahedron.

For example, if $|Y| = 2$ in the above calculations we get

$$\frac{1}{24}\left(1.2^6 + 3.2^3 + 3.2^4 + 8.2^2 + 6.2^3\right)$$

$$= \frac{1}{24}\left(64 + 24 + 48 + 32 + 48\right) = \boxed{8}$$

How about the action of R_C on the set of all 12 edges of C?

order of a	number of such	cycle type of a on edges	$\nu(a)$
1	1	1+...+1 (12 1's)	12
2_{ee}	6	2+2+2+2+2+1+1	7
2_{ff}	3	2+2+2+2+2+2	6
3	8	3+3+3+3	4
4	6	4+4+4	3

number of colorings of edges with $c = |Y|$ available colors.

$$\frac{1}{24}\left(1.c^{12} + 6.c^7 + 3.c^6 + 8.c^4 + 6.c^3\right)$$

eg. for $c = 2$

$$\frac{1}{24}\left(4096 + \underbrace{6.128 + 3.64}_{15.64} + 8.16 + 6.8\right)$$

$$= \frac{1}{24}\left(4096 + 960 + 128 + 48\right)$$

$$= \frac{1}{24}(5232) = \boxed{218}$$

How about the action of R_D on the set of all 12 faces of a dodecahedron

order of a	number of such	cycle type of a on faces	$\nu(a)$
1	1	12 1's	12
2	15	6 2's	6
3	20	4 3's	4
5	24	5+5+1+1	4
	60 ✓		

number of colorings $\frac{1}{60}\left(1.c^{12} + 15.c^6 + 20.c^4 + 24.c^4\right)$

eg for $c=2$ $\quad \frac{1}{60}(4096 + 960 + 704) = \frac{5760}{60} = \boxed{96}$