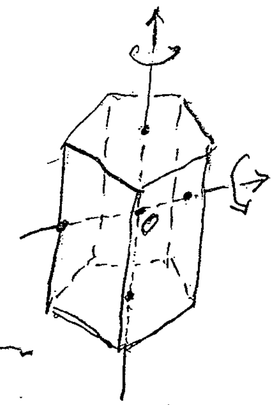
DEF For $n \geq 3$, a regular n-prism P is a polyhedron with $n+2$ faces: two regular n-gons in parallel planes perpendicular to the line through their centers and n rectangles as shown (the case $n=5$).



THM A Let P be a rectangular n-prism. With one exception noted in the proof, the the set $D_{2n} = R_P$ of all rotations which stabilize P is a subgroup of order $2n$ in R_O, where O is the point on the vertical axis midway between the top & bottom n-gon. It consists of the n rotations by angles $2\pi k/n$ with $k = 0, 1, \ldots, n-1$ about this axis, making a cyclic subgroup of order n which we call C_n (not a bad name since all cyclic groups of order n are isomorphic), and n rotations of order 2 (by angle π) about "horizontal" axes, as follows:

For n even, $n/2$ such axes through midpoints of a pair of opposite vertical edges,
and $n/2$ " " " " " " " " vertical faces.

For n odd, n such axes each through the midpoints of a vertical edge & the opposite vertical face.

Proof. A rotation stabilizing P must have one of the axes listed, all of which pass through O, so R_P is a subgroup of R_O. Each element of R_P either stabilizes the top n-gon or sends it to the bottom one — here we assume that $n \neq 4$, or if $n = 4$ then the vertical rectangles are not squares. An edge of the top n-gon can go to any edge of its image, and where this edge goes determines the rotation. Thus $|R_P| \leq 2n$. Since we have described $2n$ rotations stabilizing P, we have found them all, and $|R_P| = 2n$.

The special case excluded above, where the top & bottom n-gons and the four vertical rectangular sides are all squares is especially interesting:

THM B: Let C be a with center O. The set R_C of all rotations which stabilize C is a subgroup of order 24 of R_O consisting of the identity map of $\mathbb{R}^3$ and

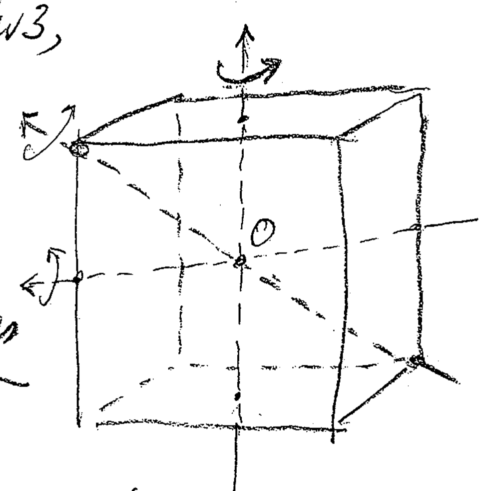
- ✓ 2 rotations of order 3 about each of the 4 lines through opposite vertices;
- 2 rotations of order 4 about each of the 3 lines through centers of opposite faces;
- ✓ 1 rotation of order 2 about each of these 3 lines;
- 1 rotation of order 2 about each of the 6 lines through midpoints of opposite edges.

Proof. Clearly each C-stabilizing rotation has axis one of the above. Since they all pass through O, R_C is a subgroup of R_O, of order

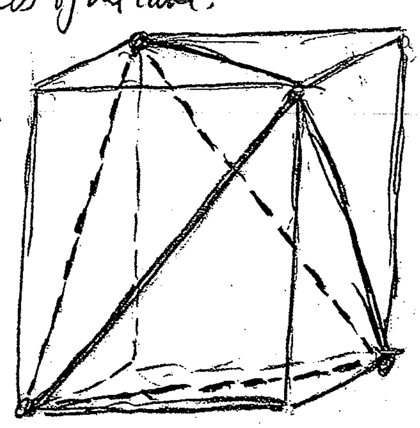
$$1 + 2\cdot4 + 2\cdot3 + 1\cdot3 + 1\cdot6 = 24.$$

As a check, there are $2\cdot12 = 24$ choices for where a given directed edge can go, and only one rotation per choice.

THM C. R_C has 4 Sylow 3-subgroups of order 3, corresponding to the 4 lines through pairs of opposite vertices; and 3 dihedral Sylow 2-subgroups of order 8, corresponding to the 3 lines through centers of opposite faces



THM D. Let T be the tetrahedron shown inscribed in the cube, its vertices 4 of the 8 vertices of the cube. (There is a second tetrahedron whose vertices are the other 4.) The subgroup of R_C which stabilizes T is R_T. It consists of 1 and the 11 checked rotations listed in THM B. The remaining 12 elements of R_C take T to the other inscribed tetrahedron.

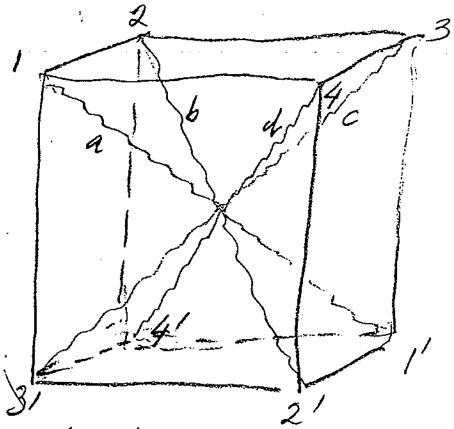


cube with one (of two) inscribed tetrahedra

THM E The group R_C is isomorphic to S_4;

Proof. R_C acts on the set of 4 long diagonals of C, i.e. (undirected) line segments connecting opposite vertices of C. This gives a homomorphism $\pi : R_C \to S_4$. Since R_C and S_4 each have order 24, to show that π is an isomorphism it suffices to show that it is surjective. Since S_4 is generated by transpositions it suffices to show that each transposition (ab) of the long diagonals a, b, c, d is achieved by a rotation of the cube.

If a & b intersect the top face in 1 & 2 then the rotation of order 2 around the line through the centers of edges 12 & 1'2' interchanges 1 & 2 and 1' & 2', therefore transposes a & b, while interchanging 3 & 3' and 4 & 4', therefore fixing c & d.



long diagonals

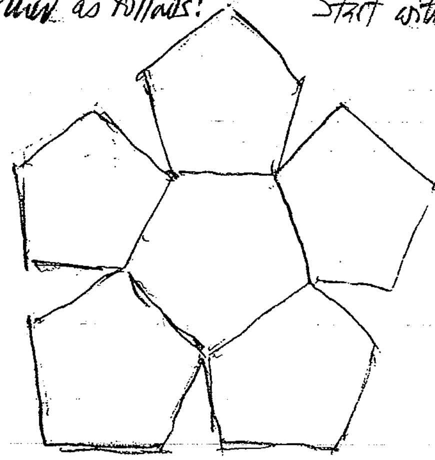
If a & b intersect the top face in 1 & 3, then rotation by π about the line through the centers of edges 13' & 31' interchanges 1 & 3' and 3 & 1', therefore transposes a & b, while interchanging 2 & 2' and 4 & 4', therefore fixing c & d.

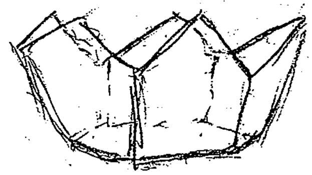
Each remaining case is similar to one of these two.

THM F The subgroup R_T of R_C (in THM D) corresponds via Thm E to A_4

Proof The four vertices of the T shown in Theorem D determine the four long diagonals of C, so the action of R_T on these vertices is the same as the action of R_T on the long diagonals of C. As permutations of the vertices of T each element $\neq 1$ of R_T is a 3-cycle or a double transposition, i.e. has cycle type 3+1 or 2+2, both of which are even permutations.

DEF. A regular dodecahedron D is a polyhedron made of 12 equal regular pentagons, i.e. 5-gons, put together as follows: Start with six pentagons, as at left.





Then fold up the outer five pentagons to make a runcible*) bowl (shown): Two such runcible bowls, put together, one on top of the other, make a dodecahedron, which has

12 (pentagonal) faces (i.e. 5 edges around each face)
30 edges
20 (trivalent) vertices (i.e. 3 edges at each vertex)

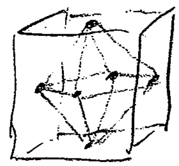
---

*) This word was coined by Edward Lear, a writer of nonsense poems in the 1800's, who used it several times for no meaningful purpose, first in The Owl and the Pussycat:

"They dined on mince, and slices of quince   -/-/, -//-/
Which they ate with a runcible spoon   --/--/--/
And hand in hand, on the edge of the sand,   -/-/, --/--/
They danced by the light of the moon."   -/--/--/

Now everyone knows that slices of quince are not to be eaten with an ordinary spoon, but rather speared by a fork. This, I believe, is the reason for the meaning which was later given to 'runcible spoon.' Now it is a synonym for spork, a hybrid of a spoon and a fork, i.e. a spoon with tines, like a fork.

DEF For each convex polyhedron in $\mathbb{R}^3$, the corresponding <u>dual polyhedron</u> is made by putting a vertex of the new at the center of each face of the old, and new edges, as many as the the old edges, eg. an ~~octahedron~~ O in a cube C (or a cube in an octahedron), and



octahedron inscribed in a cube

dual to a dodecahedron D is an <u>icosahedron</u> I, with

20 triangular faces

30 edges

12 5-valent vertices

TERMINOLOGY The group R_C is also the group stabilizing the corresponding octahedron, which is why it is called the <u>octahedral group</u>. Similarly, the group R_D is also R_I, which is why it is called the <u>icosahedral group</u>.

THM G The rotations of $\mathbb{R}^3$ stabilizing a regular dodecahedron D is a subgroup R_D of order 60 in R_O, where O is the intersection of the

6 lines connecting midpoints of opposite pentagonal faces;

15 lines connecting midpoints of opposite edges;

10 lines connecting opposite vertices.

Besides the identity element, there are

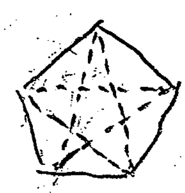
$6 \cdot 4 = 24$ elements of order 5;

15 elements of order 2;

$10 . 2 = 20$ elements of order 3;

making $1 + 24 + 15 + 20 = 60 =$ (check) $12 . 5$ rotations stabilizing the dodecahedron.

THM H The five diagonals of each pentagonal face of a regular dodecahedron D, are edges of five cubes inscribed in D.

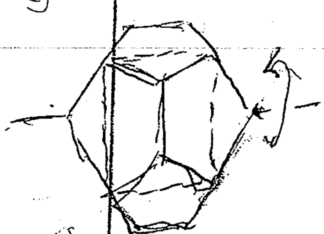


Proof Look at the model in class.

$12 \cdot 5 = 5 \cdot 12$

THM H: The group R_D is isomorphic to A_5.

Proof. R_D acts on the set of five cubes inscribed in D, so we have a homomorphism $\pi : R_D \to S_5$. Let's look at the cycle structure of the different types of elements in R_D, as permutations of the five cubes



order of element	cycle type	
1	1+1+1+1+1	
2	2+2+1	(see below)
3	3+1+1	(")
5	5	(")

2. Grabbing hold of the dodecahedron by the midpoints of two opposite edges, and rotating by $\pi = 180°$, we see immediately only one cube that is stabilized. The other four are not, and therefore make up two 2-cycles
3. Suspending the dodecahedron between two forefingers the tips of which touch opposite vertices, and rotating by $2\pi/3 = 120°$, I see the two cubes which share ~~those~~ vertices stabilized, ~~but~~ no others are, so the others must make a 3-cycle.
5. Suspending the dodecahedron between two forefingers, the tips at centers of opposite faces, we see that each of the five cubes has an edge as a diagonal of each face, and a rotation by $2\pi/5$ fixes none of these edges, so acts as a 5-cycle on the cubes

From the table above each of the permutations $\pi(a)$, for $a \in R_D$ is even, and only $a = 1$ has $\pi(a) = 1$, so $\ker \pi = 1$ and $\operatorname{im} \pi \subset A_5$, giving

$$R_D \cong R_D/1 = R_D/\ker\pi \cong \operatorname{im}\pi \subset A_5$$

Since $|R_D| = |A_5|$, we conclude that the $\subset$ is $=$, so $R_D \cong A_5$