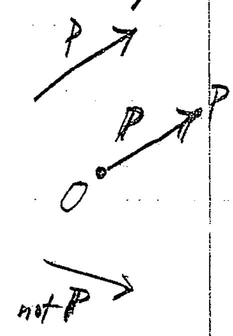
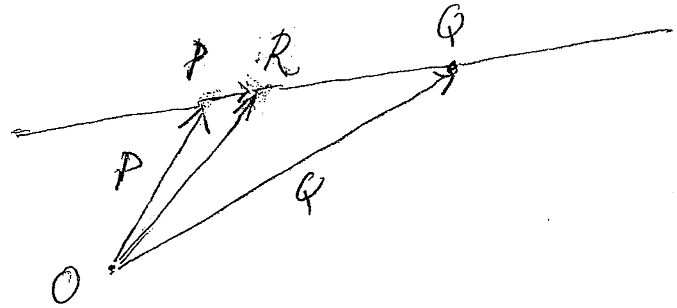
DEF Having chosen any point O in 3-space as *origin*, to each point P there is a vector from O to P, also denoted by P, the *position vector* of P (from O). Vice versa, for each vector P in space the *equal* vector (i.e. same length and same direction) which starts at O is the position vector (from O) to a point, also denoted by P. Thus we may identify points with vectors and vectors with points.



APPLICATION (Line through two points)
For any two different points P & Q in 3-space, the midpoint of the line segment from P to Q is*) $\frac{1}{2}(P+Q)$. More generally, as shown by the picture,



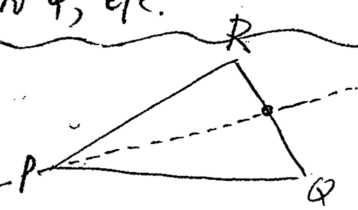
each point R on the line through P and Q has the form

$$R(t) = P + t(Q-P) = (1-t)P + tQ$$

for some real t ($t=\frac{1}{4}$ in the picture); the segment from P to Q corresponding to $0 \le t \le 1$, the part before P to $t<0$, the part after Q to $t>1$.

The case $t=\frac{1}{2}$ gives the midpoint of the segment from P to Q, $t=\frac{2}{3}$ would give the point on that segment $\frac{2}{3}$ of the way from P to Q, etc.

DEF The *medians* of a triangle are the three lines, each through a vertex and the midpoint of the opposite side.



*) meaning of is ', i.e. the position vector of the midpoint from any given origin O is the average of the position vectors from the same O to P and to Q.

THM A ($\simeq$ 2200 years old) The three medians of a triangle intersect in a point (!), which is $\frac{2}{3}$ of the way from each vertex to the midpoint of the opposite side (!).

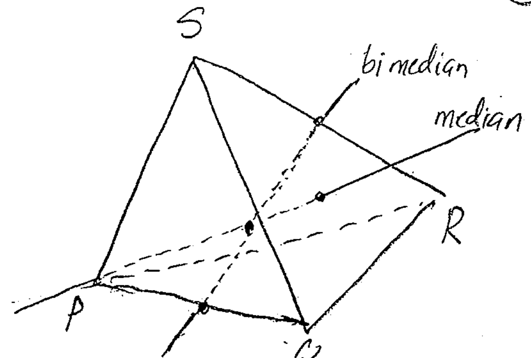
Proof Let P, Q, R be the vertices. Then

$$\boxed{\tfrac{1}{3}(P+Q+R)} = \tfrac{1}{3}P + \tfrac{2}{3}\cdot\tfrac{1}{2}(Q+R) = \tfrac{1}{3}Q + \tfrac{2}{3}\tfrac{1}{2}(P+R) = \tfrac{1}{3}R + \tfrac{2}{3}\tfrac{1}{2}(P+Q),$$

so the centroid of the triangle, given by $\square$, is on all three medians, and is $\frac{2}{3}$ ....

REMARK Though the position vector of a point depends on the choice of O, yet the midpoint of a segment (given by $\frac{1}{2}(P+Q)$) and the centroid of a triangle (given by $\frac{1}{3}(P+Q+R)$) and the centroid of a tetrahedron (given by $\frac{1}{4}(P+Q+R+S)$), regarded as points, are independent of the choice of origin.

DEF The medians of a tetrahedron $PQRS$ are the four lines, each through a vertex and the centroid of the opposite face. The bimedians are the three lines, each through the midpoints of a pair of opposite edges. In the picture, one median (through P) and one bimedian



through the midpoint of edges PQ & RS are shown. They intersect (!) In fact,

THM B ($\simeq$ 450 years old) The four medians of a tetrahedron intersect in a point (!), which is $3/4$ of the way from each vertex to the centroid of the opposite face (!). Also, the three bimedians intersect in this same point (!), which is the midpoint of each bimedian segment connecting the midpoints of a pair of opposite edges (!).

Proof Let P, Q, R, S be the vertices of the tetrahedron. Then

$$\tfrac{1}{4}(P+Q+R+S) = \tfrac{1}{4}P + \tfrac{3}{4}\tfrac{1}{3}(Q+R+S) = \text{etc} = \text{etc} = \text{etc}.$$

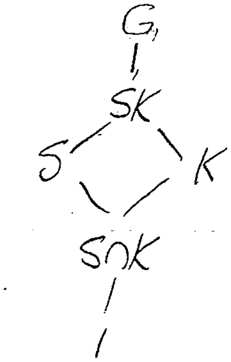
$$= \tfrac{1}{2}\left(\tfrac{1}{2}(P+Q) + \tfrac{1}{2}(R+S)\right) = \text{etc} = \text{etc}.$$

EXERCISES (mostly on §19) due Tuesday, Nov. 25

#1. Let G be a finite group, S a Sylow p-subgroup of G, K a normal subgroup of G.
Prove

(a) $S \cap K$ is a Sylow p-subgroup of K;

(b) SK/K is a Sylow p-subgroup of G/K.



#2. Let G be a group of order mp,
where m and p are primes, and $m < p$.
Let M be a Sylow p-subgroup of G and P a Sylow p-subgroup of G.
Prove that $G = M \ltimes P$ (semidirect product).
Hint: See 13.4 for the definition of <u>semidirect</u> <u>product</u>.

#3. In #2, prove that if, in addition, $p \not\equiv 1 \mod m$,
then $G = M \times P$ (direct product), and therefore G is cyclic.

#4(a) Prove that each group G of order n is cyclic, for

$$n = 15, \quad n = 33, \quad n = 35, \quad n = 51$$

(b) Are there any more $n < 60$ for which each group of order n
can be proved cyclic by application of #3.

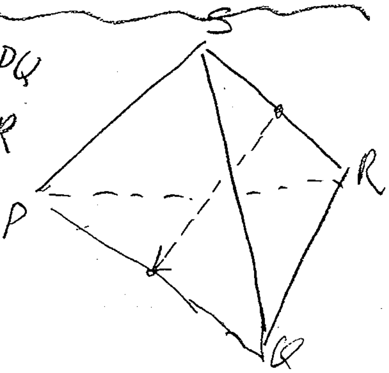
#5 For how many $n < 60$ is $n = mp$ for primes m and p with $m < p$?

#6 Let G be a finite group of order n. Prove that if G has a
subgroup K of index p, where p is the smallest prime divisor of n,
then $K \triangleleft G$.

#7 Let G be a group (not necessarily finite). Prove that if G has
a subgroup K of index 2, then $K \triangleleft G$.

THM C Two of the bimedians of a tetrahedron are perpendicular if and only if the third pair of edges of the tetrahedron have equal length.

Proof. From the midpoint of edge RS to the midpoint of edge PQ and from the midpoint of edge QS to the midpoint of edge PR are the <u>bimedian</u> <u>vectors</u>



$$\tfrac{1}{2}(P+Q)-\tfrac{1}{2}(R+S) \quad \text{and} \quad \tfrac{1}{2}(P+R)-\tfrac{1}{2}(Q+S).$$

The corresponding bimedian lines are perpendicular if and only if these vectors are perpendicular, i.e.

$$\begin{aligned} 0 &= (P+Q-R-S)\cdot(P+R-Q-S) \\ &= ((P-S)+(Q-R))\cdot((P-S)-(Q-R)) \\ &= |P-S|^2-|Q-R|^2 \qquad (\text{since } (A+B)\cdot(A-B)=|A|^2-|B|^2), \end{aligned}$$

so equivalent to the above perpendicularity is $|P-S|=|Q-R|$.

EXERCISE 1 Two of the bimedian vectors of a tetrahedron have equal length if and only if the third pair of edges of the tetrahedron are perpendicular.

DEF A tetrahedron is <u>regular</u> if all six edges have the same length.

COR. The three bimedian lines of a regular tetrahedron are perpendicular.

THM D. Let T be a regular tetrahedron, and let O be the intersection of its four medians and its three bimedians. Each nonidentity rotation of space which stabilizes T has one of the medians or bimedians as rotation axis, so the set of all rotations stabilizing T, together with the identity map 1, is a subgroup of R_O, which we denote by R_T and call the <u>tetrahedral group</u>.

$|R_T|=12$. R_T has four Sylow 3-subgroups (each of order 3) one for each median and consisting of 1 & the two rotations of order 3 about this line. R_T has only one Sylow 2-subgroup, isomorphic to $C_2\times C_2$ consisting of 1 and the rotations of order 2 about the three bimedians.

Proof. Each rotation which stabilizes T has axis which intersects the surface of T in two points. Each such point is either a vertex, or the midpoint of an edge, or the centroid of a face. Therefore each such axis is either a median or a bimedian. We see in this way $2.4=8$ rotations of order 3 and $1.3=3$ rotations of order 2. With the identity element this makes $8+3+1=12$ rotations stabilizing T.

All the rotation axes pass through O, so R_T is a subgroup of R_O.

As a check on the order of R_T, where can a face and an incident edge go? 

There are 4 faces and each has 3 edges, so there are $4.3=12$ possibilities. Knowing where a given face and incident edge goes determines the rotation, so there are at most 12 elements in R_T and we have found them all above.

Choosing the three (mutually perpendicular) bimedians as coordinate axes, the three rotations of order 2 are given by

$$a:(x,y,z)\to(x,-y,-z);\ b:(x,y,z)\to(-x,y,z);\ c:(x,y,z)\to(-x,-y,z),$$

so $ab:(x,y,z)\to a(-x,y,-z)=(-x,-y,z)=c(x,y,z)$, ie $ab=c$.

Similarly $ba=c$, $bc=cb=a$, $ca=ac=b$, $a^2=b^2=c^2=1$, so these four elements $(a,b,c,1)$ make an abelian subgroup of order 4, with no elements of order 4.

EXERCISE 2 Prove that R_T is a semidirect product of its Sylow 2-subgroup and any one of its Sylow 3-subgroups.

THM E. The tetrahedral group R_T is isomorphic to A_4.

Proof. R_T acts on the set $\{P,Q,R,S\}$ of vertices of T, giving a homomorphism π from R_T to S_4.. Lets look at the cycle type of each of the image elements:

1 : $1+1+1+1$; rotation of order 3 : $3+1$; rotation of order 2: $2+2$.

From this we observe that each image element is an even permutation, so $\operatorname{im}\pi\subset A_4$. Also we see that $\ker\pi=1$, so $R_T\cong R_T/\ker\pi\cong\operatorname{im}\pi\subseteq A_4$.

Since R_T and A_4 have the same order $12=\frac{4!}{2}$, there is equality here.