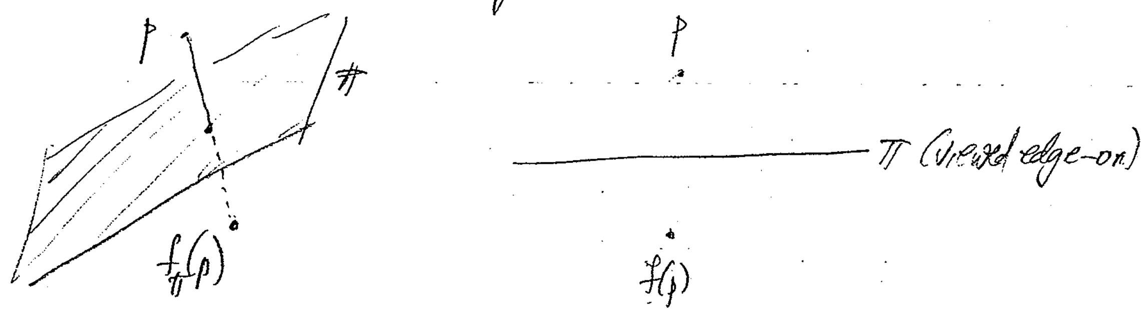
§21 Reflections and Rotations in $\mathbb{R}^3$, the Rotation Group R_0

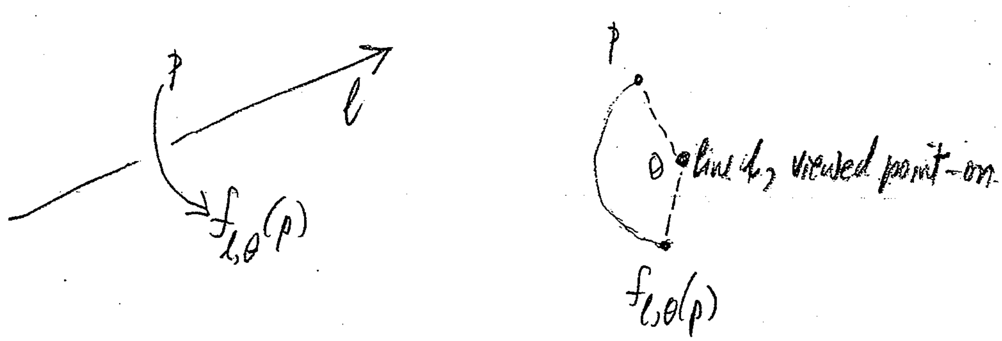
By *space* we mean the 3-dimensional space in which we live. We use intuitive language, rather than beginning with x,y,z coordinates.

DEF For each plane π in space, ***reflection*** in π is the bijection f_π of space which fixes each point of π and sends each point p not on π to the point gotten by dropping a perpendicular from p to π and going the same distance on the other side of π:

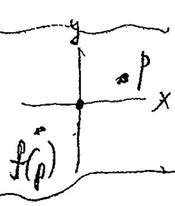


For example, if π is the xy plane, then $f_\pi(x,y,z)=(x,y,-z)$.

DEF For each directed line l in space, and each angle θ, *rotation* by θ around l is the bijection $f_{l,\theta}$ of space which fixes each point of l and rotates the rest of space rigidly around the line by an angle θ, counterclockwise when viewed *point-on*, i.e. from the arrowhead end of l



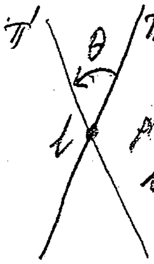
For example, if l is the z-axis and $\theta=180°$, then $f_{l,\theta}(x,y,z)=(-x,-y,z)$



EXERCISE 1 What is $f_x f_y f_z$ where f_w is $180°$ rotation about the w-axis, for $w=x,y,z$.

EXERCISE 2 Prove that $\{1, f_x, f_y, f_z\}$ is a group, isomorphic to $C_2 \times C_2$. (Here 1 is the identity map on space.)

THM A. Let π' and π be two different planes in space, not parallel, intersecting in a line l (shown at right).



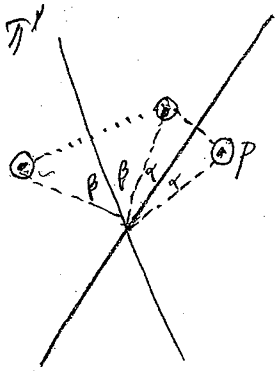
planes viewed edge-on
line viewed point-on

Then

$$\boxed{f_{\pi'} f_{\pi} = f_{l, 2\theta}},$$

where θ is the angle shown, i.e. the product of reflections in two intersecting planes is a rotation about their line of intersection by twice the angle (shown) between the planes.

Proof.



The three circled points are arbitrary p in the plane of the paper then $f_{\pi}(p)$, in this plane, then $f_{\pi'}(f_{\pi}(p))$, in this plane. You see two isosceles triangles, each with its third side perpendicularly bisected by a line, the intersection of π or π' with the plane of the paper. From the picture p is rotated around l by the sum of four angles $= 2\alpha + 2\beta = 2(\alpha+\beta) = 2\theta$.

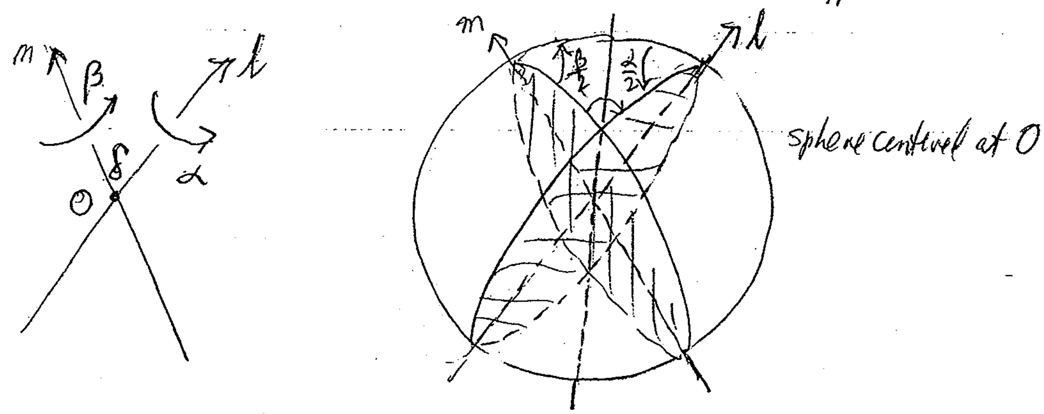
THM B For each point O in space, the set of all rotations of space about lines through O, together with the identity map 1, is a group with composition of maps as multiplication. This group is called the rotation group, and we denote it by R_O.

Proof. If $f = f_{l,\theta}$, then $f^{-1} = f_{l,-\theta}$. Thus R_O contains the inverse of each of its elements. Also $f_{l,\alpha} f_{l,\beta} = f_{l,\alpha+\beta}$, so the product of two rotations about the same line through O is also such a rotation.

To show that $f_{\ell,\alpha} f_{m,\beta}$ is a rotation $f_{n,\gamma}$ about some line n in the case $\ell \neq m$ is more fun:

If m is the same line as ℓ but oppositely directed, then $f_{m,\beta} = f_{\ell,-\beta}$, so $f_{\ell,\alpha} f_{m,\beta} = f_{\ell,\alpha-\beta}$.

The most fun case we leave for last: m is not ℓ and not opposite to ℓ.



We may suppose the angle δ shown is between 0 & $180°$ (as shown), since if not replace m by its opposite and β by $-\beta$.

The lines ℓ & m determine a plane π containing them both. If we draw ℓ & m on this paper, π is the plane of this paper. Choose planes π' and π'' so that

π' & π intersect in ℓ with angle $\alpha/2$

π & π'' intersect in m with angle $\beta/2$

The shaded discs are the intersections of π'' and π' with the sphere shown.

By THM A (applied three times),

$$f_{\ell,\alpha} \underset{(1)}{=} f_{\pi'} f_{\pi} \quad \& \quad f_{m,\beta} \underset{(2)}{=} f_{\pi} f_{\pi''},$$

so

$$f_{\ell,\alpha} f_{m,\beta} = f_{\pi'} f_{\pi} f_{\pi} f_{\pi''} = f_{\pi'} f_{\pi''} \underset{(3)}{=} f_{n,\gamma}$$

where n is the line of intersection of π' & π'' and $\gamma/2$ is the angle between π' & π''!

EXERCISE 3 For l any directed line through O, let H_l be the subgroup of R_0 consisting of all rotations about l. Prove that

$$R_0 = \bigcup_{\text{all } l} H_l .$$

and that the subgroups H_l are conjugate to each other, ie if l & m are such lines, then $H_m = gH_lg^{-1} = H_l^g$ for some $g \in R_0$.

EXERCISE 4. Prove that if G is a finite group, and H is a subgroup with the property that

$$G = \bigcup_{g \in G} H^g,$$

then $H = G$.

EXERCISE 5. The statement in EXERCISE 4 would be false if the word finite was left out. Why?