THM. A Each group $G \neq 1$ ~~with nonprime order~~ < 60 has a proper normal subgroup, i.e. a normal subgroup K other than $K = 1$ or $K = G$.

To see what has to be done, let's list the positive integers $n \leq 60$, circling the primes, the prime powers, and 1, and factoring the others into primes:

1
2
3
4
5
6 $\quad 2.\underline{3}$ ✓
7
8
9
10 $\quad 2.\underline{5}$ ✓
11
$\boxed{12 \quad 2^2.3}$
13
14 $\quad 2.\underline{7}$ ✓
15 $\quad 3.\underline{5}$ ✓
16
17
18 $\quad 2.\underline{3}^2$ ✓
19
20 $\quad 2^2.\underline{5}$ ✓
21 $\quad 3.\underline{7}$ ✓
22 $\quad 2.\underline{11}$ ✓
23
$\boxed{24 \quad 2^3.3}$
25
26 $\quad 2.\underline{13}$ ✓
27
28 $\quad 2^2.\underline{7}$ ✓
29
$\boxed{30 \quad 2.3.5}$
31
32
33 $\quad 3.\underline{11}$ ✓
34 $\quad 2.\underline{17}$ ✓
35 $\quad 5.\underline{7}$ ✓
$\boxed{36 \quad 2^2.3^2}$
37
38 $\quad 2.\underline{19}$ ✓
39 $\quad 3.\underline{13}$ ✓
$\boxed{40 \quad 2^3.5}$
41
42 $\quad 2.3.\underline{7}$ ✓
43
44 $\quad 2^2.\underline{11}$ ✓
$\boxed{45 \quad 3^2.5}$
46 $\quad 2.\underline{23}$ ✓
47
$\boxed{48 \quad 2^4.3}$
49
50 $\quad 2.\underline{5}^2$ ✓
51 $\quad 3.\underline{17}$ ✓
52 $\quad 2^2.\underline{13}$ ✓
53
54 $\quad 2.\underline{3}^3$ ✓
55 $\quad 5.\underline{11}$ ✓
$\boxed{56 \quad 2^3.\underline{7}}$
57 $\quad 3.\underline{19}$ ✓
58 $\quad 2.\underline{29}$ ✓
59
$\boxed{60 \quad 2^2.3.5}$

Proof of THM Since each subgroup of an abelian group is normal, and each group of nonprime order >1 has a (proper) subgroup of prime order (generated by any element of prime order), it follows that every finite abelian group of nonprime order >1 has a proper normal subgroup.

If $n = p^a$ with p prime and $a \geq 2$ (— in this list $4, 8, 16, 32, 9, 27, 25, 49$) then a group G of order n has a nontrivial center, by THM B in §18; if $Z(G) = G$, then G is abelian, so has a proper normal subgroup by the above, while if $Z(G) \neq G$, then $Z(G)$ itself is a proper normal subgroup.

This disposes of the circled n's.

For those of the factored n in which $n = p^a m$ with prime $p > m$ (these are checked ✓ in the list, with p underlined), a Sylow p-subgroup is normal (and proper) by Exercise 1 in §18, since there is no divisor $d > 1$ of m with $d \equiv 1 \bmod p$, such a divisor being $> p > m$.

There remain the boxed n's:

$$12, 24, 30, 36, 40, 45, 48, 56, 60$$

LEMMA Let G be a group. Then each subgroup H of G with index m contains a normal subgroup K of G with index in G dividing $m!$

$$\begin{tikzcd} G \arrow[d, dash, "m"] \\ H \arrow[d, dash] \\ K \arrow[d, dash] \\ {} \end{tikzcd}$$

Proof G acts on G/H by left translation ($g_1 \cdot gH = g_1 g H$, etc.), so we have a homomorphism of G to the symmetric group $S_{G/H}$ of order $m!$

The kernel K of this homomorphism is normal, is contained in H (since $k \circ 1H = 1H \Rightarrow k \in H$) and by the first isomorphism theorem, $G/K \cong \text{im} \subset S_{G/H}$, so $|G/K|$ divides $m!$.

COR. If a finite group G has a subgroup H of index m satisfying $m! < |G|$, then G has a proper normal subgroup.

12
24
36
48

Using the COR we may dispose of

$$12, 24, 36, 48$$

taking for H a Sylow 2-subgroup, 2-subgroup, 3-subgroup, 2-subgroup in these cases with indices

$$3 \quad 3 \quad 4 \quad 3,$$

since $3! = 6 < 12 < 24 < 48$, and $4! = 24 < 36$.

There now remain

$$30, 40, 45, 56, 60$$

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40 For $|G| = 40$, the number $s_5(G)$ must satisfy both

$$s_5(G) \mid 8 \text{ and } s_5(G) \equiv 1 \bmod 5.$$

The only divisor of 8 which is $\equiv 1 \bmod 5$ is 1, so a Sylow 5-subgroup of G is normal.

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45 For $|G| = 45$,

$$s_5(G) \mid 9 \text{ and } s_5(G) \equiv 1 \bmod 5$$

and

$$s_3(G) \mid 5 \text{ and } s_3(G) \equiv 1 \bmod 3$$

so both $s_5(G) = 1$ and $s_3(G) = 1$, showing that G has both a normal Sylow 5 subgroup and a normal Sylow 3 subgroup. It follows that G is the direct product of these, and is therefore abelian (since they are), i.e.

$\boxed{\text{Each group of order 45 is abelian !}}$.

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30 For $n=30$,

$$s_5(G) \text{ divides } 6 \text{ and is } \equiv 1 \bmod 5, \text{ so } s_5(G) = 1 \text{ or } 6$$

and

$$s_3(G) \text{ divides } 10 \text{ and is } \equiv 1 \bmod 3, \text{ so } s_3(G) = 1 \text{ or } 10$$

If $s_5(G)=1$ or $s_3(G)=1$ then G has a normal Sylow 5-or 3-subgroup.
We show that the remaining possibility i.e. $s_5(G)=6$ and $s_3(G)=10$ is impossible.

LEMMA If G has a Sylow p-subgroup of order p, then the number of elements of order p in G is $(p-1)s_p(G)$.

Proof. Any two different Sylow p-subgroups of G have intersection $=1$, as a subgroup of each, smaller than each. Therefore the number of elements of order p in G is $p-1$ (the number of elements $\neq 1$ in G_p) times the number of Sylow p-subgroups.

Returning to $n=30$ with $s_5(G)=6$ and $s_3(G)=10$, we find

$4 \cdot 6 = 24$ elements of order 5
$2 \cdot 10 = 20$ elements of order 3

44 so far, too many in a group of order 30

56 For $n=56$,

$$s_7(G) \text{ divides } 8 \text{ and is } \equiv 1 \bmod 7, \text{ so } s_7(G) = 1 \text{ or } 8.$$

If $s_7(G)=1$, then a Sylow 7-subgroup is normal, and we win.
If $s_7(G)=8$, then there are $6 \cdot 8 = 48$ elements of order 7.
A Sylow 2-subgroup has order 8 and since $8+48=56$, there can be in this case only one Sylow 2-subgroup, which must be normal, so we win again.