## §16 Actions of Groups on Sets

DEF Let $G$ be a group, and let $X$ be a set. An *action* of $G$ on $X$ is a map $G\times X\to X$, sending $(g,x)\in G\times X$ to $gx\in X$, satisfying both

✓ $gh.x = g.hx \quad \forall x\in X,\ \forall g,h\in G$

✓✓ $1.x = x \quad \forall x\in X.$ ($1$ = identity element of $G$).

EXAMPLE 1. Each group $G$ acts on itself (i.e. $X=G$) by *left translation*: here $gx$ is simply the product in $G$ of elements $g$ & $x$ in $G$, condition ✓ is the associative law for multiplication in $G$, and ✓✓ is a property of $1\in G$.

EXPONENTIAL NOTATION. For an action of a group $G$ on a set $X$, we sometimes use, instead of the above product notation, the *exponential notation* $g,x\mapsto x^g$. Rules ✓ and ✓✓, in the exponential notation, read

✓ $x^{gh} = (x^h)^g$ & ✓✓ $x^1 = x$.

EXAMPLE 2 Each group $G$ acts on itself (i.e. $X=G$) by *conjugation*: here we use exponential notation, with $x^g = gxg^{-1}$ for $g\in G$, $x\in G$. Clearly $x^1=x$ for all $x\in G$. To verify ✓✓: for all $g,h,x\in G$:

$$x^{gh} = gh.x.(gh)^{-1} = g.hxh^{-1}.g^{-1} = (x^h)^g.$$

Thus conjugation is indeed an action of $G$ on $G$.

EXAMPLE 3 Given any action of a group $G$, on a set $X$, and given $A\subset X$, we put $gA=\{ga : a\in A\}$ for each $g\in G$. Thus $gA\subset X$, and we get in this way an action of $G$ on $\mathcal{P}(X)$ (= the set of all subsets $A\subset X$). In fact, clearly $1A=A$ for each $A\subset X$, and for $g,h\in G$ and $A\subset X$,

$$gh.A = \{gh.a : a\in A\} = \{g.ha : a\in A\} = \{gb : b\in hA\} = g.hA,$$

verifying rule ✓✓ for an action of $G$ on $\mathcal{P}(X)$.

THM A. Let $G$ be a group, and let $X$ be a set.

(1) For each action of $G$ on $X$, we get a map $\pi_g : X \to X$, for each $g \in G$, defined by $\pi_g(x) = gx$. The map $\pi : g \mapsto \pi_g$ is then a homomorphism from $G$ to $S_X$.[*)]

(2) Conversely, for each homomorphism $\pi : G \to S_X$, we get an action of $G$ on $X$, defined by $gx = \pi_g(x)$.

COR 1. For each action of $G$ on $X$, each map $x \mapsto gx$ is a permutation of $X$.

Proof (1) At first all we know is that each $\pi_g$ is a map from $X$ to $X$. Equation ✓ on 16.1 may be expressed in terms of these maps as

$$\pi_{gh}(x) = \pi_g(\pi_h(x)) \quad \forall x \in X,\ \forall g, h \in G,$$

i.e. as

$$* \qquad \pi_{gh} = \pi_g \pi_h \quad \forall g, h \in G,$$

and equation ✓✓ translates to

$$** \qquad \pi_1 = 1_X .$$

Choosing $h = g^{-1}$ in $*$ and using $**$, we get

$$\pi_g \pi_{g^{-1}} = \pi_{gg^{-1}} = \pi_1 = 1_X,$$

which implies that each $\pi_g$ is surjective (since $1_X$ is). Replacing $g$ by $g^{-1}$ here gives

$$\pi_{g^{-1}} \pi_g = 1_X,$$

which implies that each $\pi_g$ is injective (since $1_X$ is). Thus each $\pi_X$ is a permutation of $X$, i.e. the map $\pi : g \mapsto \pi_g$ is a homomorphism from $G$ to $S_X$.

(2) Now let $\pi : G \to S_X$ be any homomorphism. Put $gx = \pi_g(x)$ for $x \in X, g \in G$, and consider the map $G \times X \to X$ defined by $(g, x) \mapsto gx$. From $*$ and $**$ we get ✓ and ✓✓, so this map from $G \times X \to X$ is an action of $G$ on $X$.

COR 2. (Cayley's Theorem) Each group $G$ is isomorphic to a subgroup of $S_G$.

*) Recall that $S_X$, the symmetric group on $X$, is the group of all permutations of $X$, i.e. bijections $X \to X$.

Proof. Let $\pi : G \to S_G$ be the homomorphism corresponding to the action of $G$ on $G$ by left-translation. By the 1st isomorphism theorem, $\operatorname{im}\pi$ is a subgroup of $S_G$ and

$$\checkmark \qquad \operatorname{im}\pi \cong G/\ker\pi$$

Since $g \in \ker\pi \Rightarrow g1 = 1 \Rightarrow g = 1$, we have $\ker\pi = 1$, so

$$\checkmark\checkmark \qquad G/\ker\pi \cong G.$$

From $\checkmark$ and $\checkmark\checkmark$ it follows that $G \cong \operatorname{im}\pi$, a subgroup of $S_G$.

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DEF. Given an action of a group $G$ on a set $X$, for each $x \in X$ put

$$Gx = \{gx : g \in G\}.$$
$$G_x = \{g \in G : gx = x\}.$$

$Gx$ is the <u>orbit</u> of $x$ and $G_x$ is the <u>stabilizer</u> of $x$.

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THM B For each action of a group $G$ on a set $X$, and each $x \in X$,

(1) the orbit of $x$ is a subset of $X$, and the stabilizer of $x$ is a subgroup of $G$;

(2) there is a bijection $Gx \to G/G_x$;

If, in addition, $G$ is finite, then

(3) $|Gx| = |G/G_x|$; so,

(4) $|Gx|$ divides $|G|$.

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Proof. (1) Clearly $Gx$ is a subset of $X$. Since $1x = x$ we have $1 \in G_x$. If $g$ & $h \in G_x$, then $gx = x$ and $hx = x$, so $x = 1x = h^{-1}h.x = h^{-1}x$, so $gh^{-1}.x = g.h^{-1}x = g.x = x$, showing $gh^{-1} \in G_x$. Thus $G_x$ is a subgroup of $G$.

(2) To get a bijection from $Gx$ to $G/G_x$ we first show that, for $g, g' \in G$,

$$g'x = gx \iff g'G_x = gG_x$$

In fact,

$$g'x = gx \iff g^{-1}g'x = x \iff g^{-1}g' \in G_x \iff g' \in gG_x \iff g'G_x = gG_x.$$

We may define a map $\varphi: Gx \to G/G_x$ by $\varphi(gx) = gG_x$, since $gx = g'x \Rightarrow gG_x = g'G_x$.
This map is clearly surjective. It is injective since $gG_x = g'G_x \Rightarrow gx = g'x$.
(3) and (4) follow from (2) and Lagrange's Theorem.

THM C For each action of a group $G$ on a set $X$, the orbits are the parts of a partition of $X$, called the <u>orbit space</u> of the action, and denoted $G\backslash X$.

Proof. Since each $x = 1.x \in Gx$, the orbits are nonempty and cover $X$.
It remains to show that distinct orbits are disjoint, or equivalently, nondisjoint orbits are equal. To see this, suppose $x \in Gx_1 \cap Gx_2$. Then $x = g_1x_1$ and $x = g_2x_2$ for some $g_1$ & $g_2 \in G$, so $g_1x_1 = g_2x_2$, so

$$Gx_1 = Gg_1 \cdot x_1 = G.g_1x_1 = G.g_2x_2 = Gg_2 \cdot x_2 = Gx_2.$$

DEF For each action of a group $G$ on a set $X$, and for each $g \in G$ & $x \in X$, if $gx = x$ we say $g$ <u>fixes</u> $x$, i.e. $x$ is a <u>fixed point</u> of $g$.

THM C (Cauchy–Frobenius). For each action of a finite group $G$ on a finite set $X$, the number of orbits equals the average, over all $g \in G$, of the number of fixed points of $g$.

Proof. Let $\theta(g)$ be the number of fixed points of $g$. Then

$$\sum_{g \in G} \theta(g) = \sum_{x \in X} |G_x|,$$

since each side counts the number of pairs $g, x$ with $g \in G, x \in X$ for which $gx = x$.
Dividing this equation by $|G|$ gives the average number of fixed points:

$$\frac{1}{|G|} \sum_{g \in G} \theta(g) = \sum_{x \in X} \frac{1}{|G/G_x|} = \sum_{x \in X} \frac{1}{|Gx|} = \sum_{A \in G\backslash X} \frac{1}{|A|} \sum_{x \in A} 1 = \sum_{A \in G\backslash X} 1 = |G\backslash X|.$$

DEF An action of a group $G$ on a set $X$ is <u>transitive</u> if there is only one orbit, i.e. if $X = Gx$ for some $x \in X$.

COR. In a transitive action of a finite group $G$ on a finite set $X$ with $|X|>1$, at least one element of $G$ has no fixed points.

Proof. By THM D, the average number of fixed points is 1, since there is only one orbit. The element $g=1$ fixes all points of $X$, so it has $>1$ fixed points, since $|X|>1$. To compensate, some element of $G$ must have $<1$ fixed point, i.e. no fixed points.

EXERCISE 1. Given an action of a group $G$ on a set $X$, and therefore the corresponding homomorphism $\pi : G \to S_X$, prove that

$$\ker \pi = \bigcap_{x \in X} G_x .$$

EXERCISE 2. Given an action of a group $G$ on a set $X$, and an element $x \in X$ and an element $h \in G$, prove that

$$G_{hx} = h G_x h^{-1}$$

EXERCISE 3 Let $G$ be a group, and let $G$ act on itself by conjugation. Prove that for each $x \in G$

(1) the orbit of $x$ is the set of all $gxg^{-1}$ for $g \in G$

(2) the stabilizer of $x$ is the subgroup $C_G(x) = \{g \in G \mid gx = xg\}$

In this context, the orbits are called <u>conjugacy classes</u> and the stabilizers are called <u>centralizers</u>.

EXERCISE 4. In the context of Exercise 3, prove that the kernel of the corresponding homomorphism $\pi : G \to S_G$ is the intersection of all the centralizers $C_G(x)$ for $x \in G$. It is an abelian normal subgroup called the <u>center</u> of $G$ and is denoted by $Z(G)$.