§10. Intersection and Product of Subgroups

THM A. Let H and K be subgroups of a group G. Then $H \cap K$ is a subgroup of G, i.e. "The intersection of two subgroups is a subgroup."

Proof. Since $1 \in H$ and $1 \in K$ we have $1 \in H \cap K$, so $H \cap K \neq \phi$.
If a & $b \in H \cap K$ then a & $b \in H$ and a & $b \in K$, so $ab^{-1} \in H$ and $ab^{-1} \in K$, since H & K are subgroups, i.e. $ab^{-1} \in H \cap K$. Therefore $H \cap K$ is a subgroup, by Exercise 4 on 8.2.

TERMINOLOGY. Let $m, n \in \mathbb{N}$. We say that m & n are <u>relatively prime</u> if $(m,n)=1$. Equivalently, m & n are relatively prime if they have no common prime factor, e.g. 35 and 12.

THM B. Let G be a finite group and let H & K be subgroups of G whose orders are relatively prime. Then $H \cap K = 1$.*)

Proof. Let D be the subgroup $H \cap K$. Then D is a subgroup of H, so $|D|$ divides $|H|$, by Lagrange's Theorem. Similarly $|D|$ divides $|K|$. Thus $|D|$ is a common divisor of $|H|$ and $|K|$, so $|D|$ divides $(|H|, |K|) = 1$, so $D = 1$.

NOTATION. Let H be a subgroup of G and S a subset of G. If S is a union of**) left H-cosets in G, then we write S/H for the set of those H-cosets which a contained in S. Thus $S/H \subset G/H$.
For example, since G is the union of <u>all</u> left H-cosets, G/H is the set of all left H-cosets, in agreement with earlier notation.

LEMMA If H is a subgroup of a finite group G, and S is a union of left H-cosets, then

$$|S| = |S/H|\,|H|, \quad \text{i.e. } |S/H| = |S|/|H|.$$

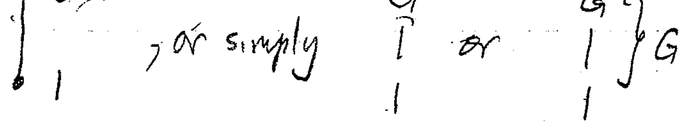
Proof S/H is a partition of S, and each part has size $|H|$.

*) By 1 here we mean the subgroup $\{1\}$.
**) some, not necessarily all

COR. If H and K are subgroups of a finite group G, then
$$|HK| = |HK/K|\,|K|, \text{ i.e. } |HK/K| = |HK|/|K|.$$

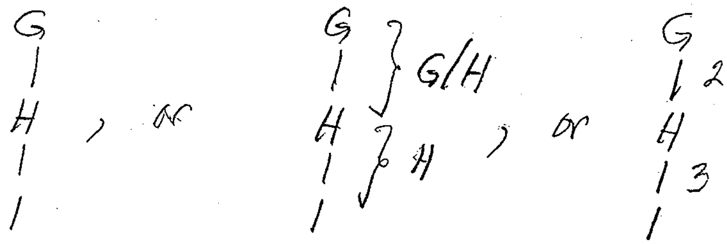
Proof. The subset $S = HK$ is a union of (some) left K-cosets.

HASSE DIAGRAMS. Now is the time to introduce "Hasse diagrams," which help to organize in pictures relationships between subsets and subgroups of groups, coset spaces and even subsets of coset spaces. A group G is pictured as



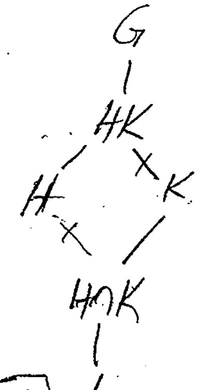
where 1 represents the trivial subgroup $\{1\}$. Sets are represented by points or line segments, with subsets below. Here $1 \subset G$.

- If H is a subgroup of G, we picture $1 \subset H \subset G$ by


if $|H| = 3$ & $|G| = 6$

Here the top segment represents the left coset space G/H.

- If H and K are subgroups of G, then the subset HK of G contains both H and K, each of which contain $H \cap K$.



Here $HK \diagdown K$ represents the subset HK/K of the left coset space G/K and $H \diagup HK$ represents the subset $H\backslash HK$ of the right coset space $H\backslash G$.

In plane geometry, opposite sides of a parallelogram are equal. Same here:

THM C(1) If H and K are subgroups of a group G, then there is a bijection $h(H \cap K) \to hK/K$ from $H/(H \cap K)$ to HK/K.

(2) Thus, if also G is finite, then $|H/H \cap K| = |HK/K|$.

Proof. To save notation, put $D = H \cap K$. We will show first that, for $h_1, h_2 \in H$,

$$* \qquad h_1 D = h_2 D \iff h_1 K = h_2 K.$$

In fact, for $h_1, h_2 \in H$

$$\begin{aligned}
h_1 D = h_2 D &\Rightarrow h_1 = h_2 d \text{ for some } d \in D && (\text{since } 1 \in D)\\
&\Rightarrow h_1 \in h_2 K && (\text{since } D \subset K)\\
&\Rightarrow h_1 K \subset h_2 K && (\text{since } KK = K)\\
&\Rightarrow h_1 K = h_2 K && (\text{since nondisjoint left cosets are equal})\\
&\Rightarrow h_1 = h_2 k \text{ for some } k \in K && (\text{since } 1 \in K)\\
&\Rightarrow h_1 \in h_2 D && (\text{since } k = h_2^{-1} h_1 \in H \cap K)\\
&\Rightarrow h_1 D \subset h_2 D && (\text{since } DD = D)\\
&\Rightarrow h_1 D = h_2 D && (\text{since nondisjoint left cosets are equal})
\end{aligned}$$

Thus, for $h_1, h_2 \in H$, we have $h_1 D = h_2 D \Rightarrow h_1 K = h_2 K$ and $h_1 K = h_2 K \Rightarrow h_1 D = h_2 D$.

Now we can define a map $\varphi : H/D \to HK/K$ by putting

$$** \qquad \varphi(hD) = hK \quad \text{for } h \in H.$$

This is a meaningful definition because, while a given left coset of D in H may be written as hD for different h's, say as $h_1 D = h_2 D$, the corresponding left cosets of K in G are the same, by $*$ above.

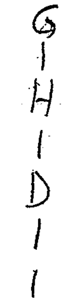
The image $\varphi(H/D)$ of H/D by φ consists of all cosets hK with $h \in H$, i.e. $\varphi(H/D) = HK/K$. Thus φ is surjective.

If, for some h_1 & $h_2 \in H$, we have $\varphi(h_1 D) = \varphi(h_2 D)$, then $h_1 K = h_2 K$, so $h_1 D = h_2 D$ by $\Leftarrow$ in $(*)$. Thus φ is injective.

It follows that $\varphi : H/D \to HK/K$ is bijective

LEMMA. If G is a finite group, H is a subgroup of G, and D is any subgroup of H, then

$$* \qquad |G/D| = |G/H||H/D|,$$



and therefore

$$** \qquad |G/H| \;\&\; |H/D| \text{ are divisors of } |G/D|.$$

Proof. The right side of $*$ is

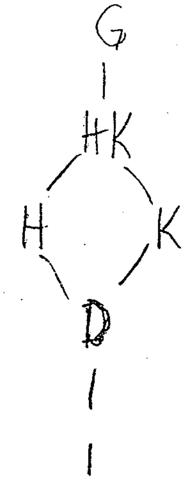
$$(|G|/|H|)(|H|/|D|) = |G|/|D|$$

after cancellation of $|H|$. Clearly $**$ follows from $*$.

THM D Let G be a finite group and let H and K be subgroups of G whose indices are relatively prime. Then $HK = G$.

Proof. Put $D = H \cap K$. Since D is a subgroup of H, $**$ in the Lemma shows that $|G/H|$ divides $|G/D|$.
Similarly, since D is a subgroup of K, $|G/K|$ divides $|G/D|$.
Since $|G/H|$ and $|G/K|$ are relatively prime, and each divides $|G/D|$, it follows by COR on 2.4 that

$$|G/H||G/K| \text{ divides } |G/D|,$$



and therefore

$$\checkmark \qquad |G/H||G/K| \leq |G/D|.$$

By $*$ in the Lemma and Theorem C,

$$\checkmark\checkmark \qquad |G/D| = |G/H||H/D| = |G/H||HK/K| \leq |G/H||G/K|$$

Combining $\checkmark$ and $\checkmark\checkmark$, we must have $=$ in both $\leq$.
In particular $|HK/K| = |G/K|$, so $HK = G$.

EXERCISE 1. Combine the Corollary on 10.2 with THM C(2) on 10.2 to prove: If H and K are subgroups of a finite group G, then

$$\boxed{|HK||H \cap K| \overset{*)}{=} |H||K|}$$

EXERCISE 2. Let $t_1 = (23)$, $t_2 = (13)$ and $r = (123)$ in S_3. Prove that

✓ $|\langle t_1\rangle\langle t_2\rangle| = 4$,

first by showing that $t_1 t_2 = r$, and second by applying $\square$ in Exercise 1. Why does ✓ show that $\langle t_1\rangle\langle t_2\rangle$ is not a subgroup of S_3? Also prove that $\langle t_1\rangle\langle r\rangle = S_3$.

EXERCISE 3. Prove that for each set $\mathcal{H}$ of subgroups of a group G, the intersection $D = \bigcap_{H\in\mathcal{H}} H$ is also a subgroup of G, and that D is a subgroup of each H in $\mathcal{H}$.

DEF Let S be any subset of a group G. Let $\mathcal{H}$ be the set of all subgroups H of G for which $S \subset H$. Put $\langle S\rangle = \bigcap_{H\in\mathcal{H}} H$. Thus $\langle S\rangle$ is the intersection of all subgroups of G which contain S. Thus $\langle S\rangle$ is a subgroup of G (by Exercise 3). $\langle S\rangle$ is called <u>the subgroup generated by</u> S.

EXERCISE 4. Prove that for each subset S of a group G, $\langle S\rangle$ is the set of all elements $g \in G$ which can be written as $g = a_1 \cdots a_n$ for some integer $n \geq 0$ and some $a_1, \ldots, a_n$ in G, with either $a_k \in S$ or $a_k^{-1} \in S$, for $k = 1, \ldots, n$. (The case $n = 0$ is to be interpreted here as $g = 1$.)

EXERCISE 5. Prove that $\langle g\rangle$ (defined in §9) $= \langle S\rangle$ with $S = \{g\}$.

*) Observe the analogy with THM H on 4.5