Introduction to Modern Algebra I
Fall, 2008
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This beautiful subject: groups, rings, fields, Galois theory, was uncovered over the last two hundred years, starting with Lagrange, Cauchy, with big progress about 1830 by Abel and Galois, and is still developing. It has a central position in mathematics.

The old question was "Can a polynomial equation of degree > 4 be solved by radicals (i.e. by an algorithm analogous to the very old formula for the roots of a quadratic)?" After many years (for this class, nine months) a short answer: "Almost never."

The style of the Modern Algebra sequence: definition, theorem, proof, example, is somewhat more abstract than that of e.g. the calculus courses, where the goal appears to be to learn algorithms for the solution of standard problems. Actually the goals are somewhat similar, to learn the theory, but here it is approached more directly, as for example in the analogous Modern Analysis course. In speed this course is a couple of years slower than the first year graduate courses.

Upper level undergraduate courses which have this course or Modern Algebra I and II as prerequisite: Algebraic Number Theory, Algebraic Curves, Representations of Finite Groups, Introduction to Topology, Knot Theory, Introduction to Algebraic Topology.

Prerequisites for this course: Calculus IV and Linear Algebra, or Honors Math A-B.

There will be two midterms (with no makeups) on Thursday, October 2 and Thursday, November 6. The final exam is tentatively (almost surely) scheduled for Tuesday, December 16 from 4:10-7.

There is no text, but I will hand out notes each lecture. Each set of notes contains exercises. The solutions of each week's exercises are due the following Tuesday, in class.

The two midterms and the exercise solutions each count 1/5 and the final exam 2/5 toward the final grade.

My office hours: M & W from 4 to 5 in 411 Math.

The topics of Modern Algebra I: Brief introductions to set theory and number theory through the unique factorization theorem and Chebychev's estimate for the number of primes < x . For the rest of the Fall term, groups: subgroups and coset spaces, isomorphism theorems, finite abelian groups and their characters, actions of groups, p-groups, Sylow subgroups, symmetric group and the rotation group, counting colorings of the regular tetrahedron and the cube.

Modern Algebra II starts with rings, then commutative rings and ideals, integral domains, fields, e.g. the the ring of algebraic integers, and the field of algebraic numbers. Unique factorization in polynomial rings in one variable with coefficients in a field or in Z. A bit of linear algebra, then for the rest of the term, Galois Theory from the top down and from the bottom up, i.e.Galois groups as groups of automorphisms of a field, and their action on the set of roots of a polynomial. Finite fields. Galois' solvability theorem, Dedekind's theorem relating splitting type to cycle type. Van der Waerden's 1935 theorem that the Galois group of a monic polynomial over Z is almost always the symmetric group, so almost never are such polynomials of degree > 4 solvable by radicals.

Introduction to Modern Algebra I (Fall 2008)

1. Divisors, greatest common divisors, prime numbers
2. Unique factorisation and least common multiple in N
3. Chebychev's estimate for pi(x)
4. Algebra of sets
5. Algebra of maps
6. Inverse maps, partitions
7. Monoids, groups
8. Products of subsets, translation, subgroups, Lagrange's theorem
9. Rules for powers, orders of elements, cyclic groups
10. Intersection and product of subgroups
11. Normal subgroups, factor groups, first isomorphism theorem
12. More isomorphism theorems, solvable groups
13. Direct and semidirect products
14. Characters of finite abelian groups, extensions and othhogonality
15. Splitting, duality and basis theorems for finite abelian groups
16. Actions of groups on sets
17. Number of conjugacy classes, inner automorphisms
18. Groups of prime power order, Sylow's theorems
19. Some applications of Sylow's theorems to groups of order < 60
20. Symmetric and alternating groups
21. Reflections and rotations in R^3, the rotation group
22. Tetrahedra, the tetrahedral group
23. Dihedreal, octahedral icosahedral group
24. Counting orbits of colorings
25. Legendre's proof of Euler's polyhedron formula, regular polyhedra.

ALGEBRA BOOKS & HOW TO LEARN TO PROVE

Here are some books on algebra, roughly at the level of this course:

S. Lang, Undergraduate Algebra, Springer 2005
A. Clark, some title like Algebra , Dover paperback
D. Dummit, R. Foote, B. Holland, Abstract Algebra, Wiley, 1999
I. Herstein, Topics in Algebra, Wiley, 1999
J. Fraliegh, A First Course in Abstract Algebra, Addison-Weskley, 2003
M. Artin, Algebra, Prenticsd Hall, 1991

On writing proofs, there is a very nice course, Math 2000, taught by Dusa McDuff. The text, whose name I forget, should be in the Columbia Book Store, or if they are out, they will still know its name. Even if you are not taking that course you might sit in.

The Algebra books listed above, aimed at people who are beginning Modern Algebra, show by example what proofs look like after they are written, which is not to say the process that leads to a proof, including the revision process which gets you from the first glimmering of a potentially useful idea to a polished version of a proof. That one picks up by practice, and by reading lots of proofs. Sometimes it is useful to revise something you have read, to improve the exposition. This also helps put it in long term memory. Writing a proof is not altogether unlike how one writes an essay or a story or a poem or song, except proofs at the level of this course are in their final form generally shorter than the above, and easier to write than good songs.

About the book list: I don't follow any of them but have picked up expository ideas from Lang and Clark and other places. The last three have been used recently at Columbia as texts for Modern Algebra, Fraliegh most recently and also currently in Prof. Bayer's off-track (I in Spring, II in Fall) course. So Fraliegh should be in the bookstore. There is also a book by Beachy and Blair (check it out on amazon.com or simply google Beachy and Blair), which from what I have seen of it on the net, is very well written. It is used in one of the three sections of UCal Berkeley's Modern Algebra course.

Besides books, there are free notes on the net, e.g.:

If you google Milne notes, you will come to J. S. Milne's notes for several courses he has taught at the University of Michigan. Most immediately relevant is Group Theory, which you can download. It is for a first year graduate course he taught at University of Michigan, so somewhat but not very much above the level of our course, and very well done.