Modern Algebra II, fall 2020, Instructor M.Khovanov

Homework 7, due Wednesday October 28.

Read Rotman Irreducible Polynomials (pages 38-43) and Splitting Fields (pages 50-57) sections or follow lecture notes.

1-4. (20 points each) Solve exercises 63, 64, 65, 66 in Rotman, page 43. Use the result of exercise 63 to solve exercise 64.

5. (10 points) For which of the following polynomials can one use the Eisenstein criterion to conclude that they are irreducible over $\mathbb{Q}$?

$$x^9 - 55, \quad x^7 - 4x^4 + 6, \quad x^5 + 3x^2 + 3x + 9,$$
$$x^4 - 15x^2 + \frac{25}{2}x - 20, \quad x^5 - \frac{3}{5}x^4 + 9x - 3.$$

6. Recall that a field F of characteristic p is called *perfect* if any element $a \in F$ has a p-th root in F, that is there exists $b \in F$ such that $b^p = a$.

(a) (10 points) Following the proof given in class, explain why any finite field F is perfect.

(b) (10 points) Let field F has characteristic p and consider the field $F(t)$ of rational functions in t with coefficients in F. Any element of $F(t)$ has the form $f(t)/g(t)$ subject to the usual manipulation and cancellation rules, where polynomials f and g have coefficients in F.

Show that t has no p-th root in $F(t)$. Hint: assume otherwise, $b^p = t$ for some $b = f(t)/g(t)$. Find a polynomial relation on f and g. Use unique factorization in the polynomial ring $F[t]$ to get a contradiction.

(c) (optional) Can you use part (b) to show that $x^p - t$ is irreducible over $F(t)$? (Note that having no roots is not enough for irreducibility when the degree is greater than 3.) Hint: Enlarge $F(t)$ to a field where this polynomial fully factors. This shows that $x^p - t$ is inseparable (not separable) over $F(t)$. What is the formal derivative D of this polynomial? How does $x^p - t$ factor in its splitting field?