Modern Algebra II, fall 2020, Instructor M.Khovanov

Homework 12, due Wednesday December 9.

Review solvable groups following Rotman (2nd part of Appendix B) or any other source.

1. (30 points) In lecture 19, we started with a characteristic 0 field F and first passed to the splitting field K of $x^n - 1$, adding all n-th roots of unity. Then we formed the splitting field E of the polynomial $x^n - u$, for some $u \in F$. There is a chain of inclusions $F \subset K \subset E$. Review our construction and write in your own words and explanations why the Galois group $G = Gal(E/F)$ is a subgroup of the group of affine symmetries $A = Aff(\mathbb{Z}/n)$. Recall that the latter group A consists of "affine" transformations of $\mathbb{Z}/n$. These are transformations that take $c \in \mathbb{Z}/n$ to $ac + b$ for fixed a, b, with invertible $a \in (\mathbb{Z}/n)^*$ and $b \in \mathbb{Z}/n$. Such a transformation is associated to a pair (a, b) as above. Explain why G is a subgroup of A (examine the action of G on the set of roots of $x^n - u$ and match this action to the action of A on $\mathbb{Z}/n$).

2. (15 points) Let H be a normal subgroup of G. Prove that G is solvable iff both H and G/H are solvable. (Alternatively, you can review the proof in the lecture and write it down in your own words.)

3. (15 points) Show that the group A in Problem 1 is solvable. This implies the Galois group G in that problem is solvable.

4. (20 points) (a) Suppose that a finite group G has order p^n, for a prime p. Recall a theorem from Modern Algebra I that any such group has nontrivial center $Z(G)$. Explain why the center $Z(K)$ of any group K is a normal solvable subgroup of K. Now use induction on n to show that G is solvable (you'll need to use its center $Z(G)$ somehow).

(b) List all groups of order 8 that you know. Which ones of them are solvable?

5. (20 points) (a) In class we studied the ring Sym of symmetric functions, a subring of $F[\alpha_1, \dots, \alpha_n]$. Consider the case $n = 2$ and the power sum $p_m = \alpha_1^m + \alpha_2^m$. Prove by induction on m that p_m belongs to the ring $S = F[s_1, s_2]$ generated by elementary symmetric functions $s_1 = \alpha_1 + \alpha_2$ and $s_2 = \alpha_1\alpha_2$. (Hint: relate p_{m+1} to $p_m s_1$.)

(b) Which of the following polynomials in $F[x_1, x_2]$ are symmetric?

$$(x_1 + x_2)^3 + 2x_1x_2, \ (x_1 - x_2)^5, \ x_1^3x_2^2 + x_2^3x_1^2?$$

(c) (optional) Show that, when $n = 2$, ring S is all of Sym. (This is true for any n but takes more effort to prove.)