Modern Algebra II, Fall 2020, Instructor M. Khovanov

Homework 1, due Wednesday September 16 by 8pm.

Use notes for lecture 1 or read Rings section in Rotman (we have not yet defined polynomial rings, but it's not a difficult concept to understand). Notice that in Rotman all rings are commutative starting from page 8. For more examples, you can read Judson Section 16.1 and Howie Section 1.1.
In this homework set, rings are not necessarily commutative and they contain 1. Note that $1 = 0$ in R if and only if R is the zero ring.

1. We say that $S \subset R$ is a subring of R if S is an abelian group, contains $1 \in R$ and closed under multiplication. Show that this definition is equivalent to the one in Rotman, page 12: A subring of a ring R is a subset S of R which contains 1 and which is closed under subtraction and multiplication.

2. (a) Prove that $\mathbb{Z}\left[\frac{1}{n}\right] = \{\frac{m}{n^k} | m \in \mathbb{Z}, k \in \mathbb{N}\}$ is a subring of $\mathbb{Q}$. Here we fix $n > 1$.

(b) Can you determine for which n and ℓ rings $\mathbb{Z}\left[\frac{1}{n}\right]$ and $\mathbb{Z}\left[\frac{1}{\ell}\right]$ coincide (as subrings or $\mathbb{Q}$)? Try $n = 2, \ell = 4$ or $n = 3, \ell = 6$ or $n = 6, \ell = 12$.

(c) (optional) Are there subrings of $\mathbb{Q}$ other than $\mathbb{Z}$ and $\mathbb{Z}\left[\frac{1}{n}\right]$?

3. (a) Prove that the intersection $R_1 \cap R_2$ of two subrings of R is a subring of R. More generally, can you prove that the intersection of any collection of subrings of R is a subring of R?

(b) (optional) Find the intersection of subrings $\mathbb{Z}\left[\frac{1}{n}\right]$ and $\mathbb{Z}\left[\frac{1}{\ell}\right]$ of $\mathbb{Q}$. Here n, ℓ are natural numbers, $n, \ell > 1$.

4. Which of the following subsets of $\mathbb{R}$ with its usual ring structure (addition, multiplication) are subrings of $\mathbb{R}$? Please give brief justifications or explanations.
(a) $\mathbb{Z}$,
(b) $3\mathbb{Z} = \{3n | n \in \mathbb{Z}\}$,
(c) $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \ : \ a, b \in \mathbb{Z}\}$,
(d) $\mathbb{Q}[\sqrt{5}] = \{a + b\sqrt{5} \ : \ a, b \in \mathbb{Q}\}$,
(e) $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \ : \ x \geq 0\}$,

(f) $R = \{a + b\sqrt[3]{3} \ : \ a, b \in \mathbb{Q}\}$,
(g) $\mathbb{Q}[\sqrt[3]{3}] = \{a + b\sqrt[3]{3} + c\sqrt[3]{9} : a, b, c \in \mathbb{Q}\}$.

5. (Ring of functions on a set.) If R is a ring and S is a set, let R^S denote the set of all functions $S \longrightarrow R$ from the set S to R. Such a function f assigns an element $f(s) \in R$ to each element $s \in S$. Equip R^S with the operations of pointwise addition and multiplication; that is, if $f, g : S \longrightarrow R$, then

$$f + g \ : \ s \mapsto f(s) + g(s)$$

and

$$fg \ : \ s \mapsto f(s)g(s).$$

Prove that R^S is a ring. (Hint: For the zero element 0 in R^S choose the constant function c with $c(s) = 0$ for all $s \in S$. For the identity element, use the constant function e with $e(s) = 1$ for all $s \in S$.

6. (a) Give an example of an object X whose symmetry group $\mathrm{Sym}(X)$ is the cyclic group C_n of order n. (Hint: the symmetry group of a regular n-gon is the dihedral group D_n. Can you break the symmetry down to C_n?) This exercise is purposely vague, not specifying what we mean by an object. You are free to use graphs, polygons, higher-dimensional shapes, sets with additional structure, etc. as objects.

(b) (optional) Can you give an example of an object with the symmetry group $\mathbb{Z}$? With the symmetry group $C_n \times C_m$?