① Convert to an augmented matrix

$$\left[\begin{array}{cc|c} 2 & 3 & 2 \\ 3 & 5 & -1 \\ -1 & -2 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} -1 & -2 & 3 \\ 3 & 5 & -1 \\ 2 & 3 & 2 \end{array}\right](-1)\cdot \Longrightarrow \left[\begin{array}{cc|c} 1 & 2 & -3 \\ 3 & 5 & -1 \\ 2 & 3 & 2 \end{array}\right]\begin{array}{l} \\ (-3)\mathrm{I} \\ (-2)\mathrm{I} \end{array} \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 2 & -3 \\ 0 & -1 & 8 \\ 0 & -1 & 8 \end{array}\right]\begin{array}{l} \\ \\ -\mathrm{II} \end{array} \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & -3 \\ 0 & -1 & 8 \\ 0 & 0 & 0 \end{array}\right]\begin{array}{l} \\ (-1) \\ \\ \end{array} \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & -3 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{array}\right]\begin{array}{l} -2(\mathrm{II}) \\ \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{cc|c} 1 & 0 & 13 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{array}\right]$$

there is a unique solution $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ -8 \end{pmatrix}$

(2)

$$A = \begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 3 & 2 \end{bmatrix} \quad 3\times 2 \text{ matrix} \qquad B = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix} \quad 2\times 2 \text{ matrix}$$

$$AB = \begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0-4 & 0+0 \\ -2-2 & -1+0 \\ 6-4 & 3+0 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ -4 & -1 \\ -2 & 3 \end{bmatrix}$$

AA is not defined, since A is not a square matrix

BA is not defined $(2\times 2)(3\times 2)$ not defined, dimensions don't match

BB is defined

$$BB = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 4-2 & 2+0 \\ -4-0 & -2+0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -4 & -2 \end{bmatrix}$$

(3) $A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$ $\qquad \mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2$

Im A consists of vectors $A\vec{x}$

$$\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1\begin{bmatrix} 2 \\ -4 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ -6 \end{bmatrix} = (2x_1+3x_2)\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

vector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ spans the image of A. Also, any vector of the form $\begin{pmatrix} t \\ -2t \end{pmatrix}$, $t \in \mathbb{R}$, $t \neq 0$ spans the image of A.

kernel of A consists of solutions to $A\vec{x} = 0$. This is a homogeneous SLE.

$$(A|\vec{0}) = \left(\begin{array}{cc|c} 2 & 3 & 0 \\ -4 & -6 & 0 \end{array}\right) \xrightarrow{(1/2)} \left(\begin{array}{cc|c} 1 & 1.5 & 0 \\ -4 & -6 & 0 \end{array}\right)_{+4(I)} \longrightarrow \left(\begin{array}{cc|c} 1 & 1.5 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

kernel consists of $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that $1 \cdot x_1 + 1.5x_2 = 0$

$x_2 = t$, $x_1 = -\frac{3}{2}t$ $\qquad \begin{pmatrix} -\frac{3}{2}t \\ t \end{pmatrix}$.

take $t=1$. Vector $\begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$ spans the kernel of A.

(4) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \end{bmatrix}$ To find A^{-1}, augment A with the identity matrix and manipulate rows to convert A to I.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ +I \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} -2(III) \\ +III \\ \\ \end{array} \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Multiply out to check

$$A A^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1+0+2 & 0+0+0 & -2+0+2 \\ 0+1-1 & 0+1+0 & 0+1-1 \\ 1+0-1 & 0+0+0 & 2+0-1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

5 (20 points). For each statement, circle whether it's true or false. Give very brief (extremely brief is ok) justification for each answer, either next to it or in the blue book. Each question is worth 3 points (2 points are given for free to complete $6 \times 3 = 18$ to 20 points).

The matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is in reduced row-echelon form. True False

there is a non-zero number above one of the pivots

The matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix}$ is in reduced row-echelon form. True False

There is a number different from 1 in one of the pivot positions

Any system of two linear equations with three variables has at least one solution. True False

$\begin{cases} x+y+z=0 \\ x+y+z=1 \end{cases}$ has no solutions

The matrix $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ has rank 2. True False

$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ 1 pivot $\Rightarrow$ rank $= 1$

Matrix $\begin{bmatrix} 2 & 1 \\ 1 & k \end{bmatrix}$ is invertible for infinitely many values of k. True False

$\det\begin{pmatrix} 2 & 1 \\ 1 & k \end{pmatrix} = 2k-1$

invertible $\Leftrightarrow \det \neq 0 \Leftrightarrow k \neq \frac{1}{2}$

there are infinitely many k such that $k \neq \frac{1}{2}$

A 3×5 matrix can have rank 4. True False

rank $\leq$ # rows, # columns