Homework 9 Section 5.2

2 (+ #16)

$\vec{v}_1 = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$ $|\vec{v}_1|^2 = 6^2 + 3^2 + 2^2 = 49$ $|\vec{v}_1| = 7 \Rightarrow$

$\vec{u}_1 = \frac{1}{7}\vec{v}_1 = \begin{bmatrix} 6/7 \\ 3/7 \\ 2/7 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$

$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix} - \frac{1}{49}(12 - 18 + 6)\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix} - 0 = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$

↑ this coefficient is 0 since $\vec{u}_1$ and $\vec{v}_2$ are orthogonal!

$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{|\vec{v}_2^{\perp}|} = \frac{1}{7}\begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$

$r_{11} = |\vec{v}_1| = 7$ $r_{12} = \vec{u}_1 \cdot \vec{v}_2 = 0$ $r_{22} = |v_2^{\perp}| = 7$

$R = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$ $Q = \frac{1}{7}\begin{bmatrix} 6 & 2 \\ 3 & -6 \\ 2 & 3 \end{bmatrix}$

$M = QR = \underbrace{\frac{1}{7}\begin{bmatrix} 6 & 2 \\ 3 & -6 \\ 2 & 3 \end{bmatrix}}_{Q.} \cdot \underbrace{\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}}_{R}$

#4 (+#18)

$$\vec{v}_1 = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 25 \\ 0 \\ -25 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \quad \vec{u}_1 = \frac{\vec{v}_1}{|\vec{v}_1|} = \frac{1}{5}\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} 25 \\ 0 \\ -25 \end{bmatrix} - \frac{1}{25}(100-75)\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ 0 \\ -28 \end{bmatrix} = 7\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{|\vec{v}_2^{\perp}|} = \frac{1}{5}\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$$

$$\vec{v}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)u_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} - \frac{1}{25} \cdot 0 \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} - \frac{1}{25} \cdot 0 \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

both products are 0

$$\vec{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{u}_1 = \frac{1}{5}\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \quad \vec{u}_2 = \frac{1}{5}\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} \quad \vec{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$r_{11} = |\vec{v}_1| = 5 \quad r_{12} = \vec{u}_1 \cdot \vec{v}_2 = 5 \quad r_{13} = \vec{u}_1 \cdot \vec{v}_3 = 0$

$r_{22} = |\vec{v}_2^{\perp}| = 35 \quad r_{23} = \vec{u}_2 \cdot \vec{v}_3 = 0$

$r_{33} = |\vec{v}_3^{\perp}| = 2$

$$Q = \frac{1}{5}\begin{bmatrix} 4 & 3 & 0 \\ 0 & 0 & -5 \\ 3 & -4 & 0 \end{bmatrix} \quad R = \begin{bmatrix} 5 & 5 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$M = QR.$

#32 Find an orthonormal basis of the plane $x_1 + x_2 + x_3 = 0$.

Can take $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$|\vec{v}_1|^2 = 2 \qquad |\vec{v}_1| = \sqrt{2} \qquad \vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\,\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot 1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1/2 \\ +1/2 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad |\vec{v}_2^{\perp}| = \frac{1}{2}\sqrt{1+1+4} = \frac{\sqrt{6}}{2}$$

$$\vec{u}_2 = \cancel{|\vec{v}_2^{\perp}|}\ \frac{\vec{v}_2^{\perp}}{|\vec{v}_2^{\perp}|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Orthonormal basis: $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

Section 5.3 #36.

$$\begin{bmatrix} 2/3 & 1/\sqrt{2} & a \\ 2/3 & -1/\sqrt{2} & b \\ 1/3 & 0 & c \end{bmatrix}$$

conditions: $|\vec{v_i}|=1$ $\quad v_i \cdot v_j = 0 \quad i \neq j$.

$$\left.\begin{aligned} &\tfrac{1}{3}(2a+2b+c)=0 \Rightarrow 2a+2b+c=0 \\ &\tfrac{1}{\sqrt{2}}(a-b)=0 \Rightarrow a=b \end{aligned}\right\} \Rightarrow \begin{aligned} a&=b \\ c&=-4b \end{aligned}$$

$$\vec{V_3} = \begin{pmatrix} a \\ a \\ -4a \end{pmatrix} \qquad |\vec{V_3}|=1 \Rightarrow a^2+a^2+16a^2=1 \Rightarrow a^2=\frac{1}{18} \quad a=\frac{1}{3\sqrt{2}}$$

$\Rightarrow$ the matrix is

$$\begin{bmatrix} 2/3 & 1/\sqrt{2} & \frac{1}{3\sqrt{2}} \\ 2/3 & -1/\sqrt{2} & \frac{1}{3\sqrt{2}} \\ 1/3 & 0 & \frac{-4}{3\sqrt{2}} \end{bmatrix}$$

#54) Dimension of the space of all skew-symmetric $n\times n$ matrices.

$A = -A^T$. $\Rightarrow$ all diagonal entries are 0.

Off-diagonal $a_{ij} + a_{ji} = 0$

$\Rightarrow$ uniquely determined by a_{ij} for $i<j$.

$i=1$ $j=2,\ldots n$ $n-1$ choices

$i=2$ $j=3,\ldots n$ $n-2$ choices

$\vdots$

$i=n-1$ $j=n$ 1 choice.

Total # of such pairs is $(n-1)+(n-2)+\cdots+1 = \frac{n(n-1)}{2}$.

The dimension is $\frac{n(n-1)}{2}$.