Homework 8 | Section 4.3

#6 $T(M) = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} M \qquad U^{2\times 2} \to U^{2\times 2}$

basis B: $f_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad f_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f_3 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

$T(f_1) = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = f_1$

$T(f_2) = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = f_2$

$T(f_3) = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 0 & 3 \end{pmatrix} = 3f_3$

Matrix of T in basis B is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

T is an isomorphism on $U^{2\times 2}$.

It has rank 3.

#18 $T(z) = (2+3i)z$ $\quad \mathbb{C} \xrightarrow{T} \mathbb{C}$

basis $(1, i)$ of $\mathbb{C}$.

$T(1) = 2+3i$, $T(i) = 2i + 3i^2 = -3+2i$

Matrix of T in basis $(1, i)$ is

$$A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}.$$

T is invertible, the inverse is multiplication by $(2+3i)^{-1} = \frac{2-3i}{2^2+3^2} = \frac{2-3i}{13} = \frac{2}{13} - \frac{3}{13}i$

$$A^{-1} = \frac{1}{13}\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}.$$

rank of A is 2.

#34 $T(M) = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} M - M \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$ $\quad \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$

basis $\mathcal{B} = \left(\underset{e_{11}}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{e_{12}}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{e_{21}}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{e_{22}}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right)$

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

$$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 0 & 0 \end{pmatrix} = -3\,e_{12}$$

$$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} = 3\,e_{21}$$

$$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} = 0$$

Matrix of T in this basis is

$$\begin{array}{c c} & \begin{matrix} e_{11} & e_{12} & e_{21} & e_{22} \end{matrix} \\ \begin{matrix} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} = A$$

A is not invertible.

$\ker A$ is spanned by e_{11}, e_{22}. $\dim \ker A = 2$

$\operatorname{im} A$ is spanned by e_{12}, e_{21} $\quad \dim \operatorname{im} A = 2$

$\operatorname{rank} A = \dim(\operatorname{im} A) = 2.$

42) a) $B = \left(\underset{f_1}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \underset{f_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{f_3}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \right)$

standard basis $\mathcal{U} = \left(\underset{u_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{u_3}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right)$

$f_1 = u_1 + u_3$, $f_2 = u_2$, $f_3 = u_1 - u_3$.

Change of basis from B to $\mathcal{U}$ is

$$S = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \end{array} \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \end{array}$$

b). $T(M) = M \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} M \qquad U^{2\times 2} \to U^{2\times 2}$

$T(f_1) = 0$, $T(f_2) = 0$, $T(f_3) = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} = 4f_2$

$T(u_1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2u_2$, $T(u_2) = T(f_2) = 0$, $T(u_3) = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} = -2u_2$

Matrix of T in basis B

$$B = \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

Matrix of T in standard basis $\mathcal{U}$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$SB = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow SB = AS.$

$$AS = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

#42 c)

Change of basis matrix from $\mathcal{U}$ to $\mathcal{B}$ is the inverse of S!

Express u's in terms of f's.

$f_1 = u_1 + u_3,\ f_2 = u_2,\ f_3 = u_1 - u_3 \Rightarrow$

$f_1 + f_3 = 2u_1,\ f_2 = u_2,\ f_1 - f_3 = 2u_3 \Rightarrow$

$u_1 = \frac{1}{2}f_1 + \frac{1}{2}f_3,\ f_2 = u_2,\ u_3 = \frac{1}{2}f_1 - \frac{1}{2}f_3.$

The matrix is
$$\begin{array}{ccc} u_1 & u_2 & u_3 \end{array}$$
$$\begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{pmatrix}$$

this matrix is S^{-1}

Section 5.1 #28 $v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

v_1, v_2, v_3 are pairwise orthogonal $v_i \cdot v_j = 0$ for $i \neq j$.

rescale them to unit vectors $|v_i| = 2$ for $i = 1, 2, 3$

$$u_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad u_2 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \quad u_3 = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\text{proj}(v) = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 + (v \cdot u_3)u_3$$

$$= \frac{1}{2}u_1 + \frac{1}{2}u_2 + \frac{1}{2}u_3 = \frac{1}{2}(u_1 + u_2 + u_3) = \frac{1}{4}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}\right)$$

$$= \frac{1}{4}\begin{pmatrix} 3 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/4 \\ -1/4 \\ 1/4 \end{pmatrix}$$