Section 3.4 # 24

$$A = \begin{bmatrix} 13 & -20 \\ 6 & -9 \end{bmatrix} \quad \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

matrix $S = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$\det S = 2 \cdot 3 - 5 \cdot 1 = 1$

find the inverse of S

$$S^{-1} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

The matrix B of T in basis $\mathfrak{B}$ is the S-conjugate of matrix A.

$$B = S^{-1} A S = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 13 & -20 \\ 6 & -9 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 39-30 & -60+45 \\ -13+12 & 20-18 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 9 & -15 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 18-15 & 45-45 \\ -2+2 & -5+6 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

#42) Find a basis $\mathfrak{B}$ of $\mathbb{R}^3$ such that the matrix T of reflection about the plane $x_1 - 2x_2 + 2x_3 = 0$ is diagonal.

Solution the plane consists of all vectors $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ that are orthogonal to the vector $\vec{v} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$. (normal vector to the plane)

Under the reflection, $\vec{v}$ goes to $-\vec{v}$, $T(\vec{v}) = -\vec{v}$.

We can take $\vec{v}$ as one of the basis vectors $\vec{v}_1 = 1\vec{v} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.

Every vector in the plane $x_1 - 2x_2 + 2x_3 = 0$ is fixed by T.

If we select a basis $\vec{v}_2, \vec{v}_3$ of the plane, add $\vec{v}_1$ as above, the matrix of T in this $\underbrace{\text{basis}}_{\vec{v}_1, \vec{v}_2, \vec{v}_3}$ is $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, diagonal

the plane is the kernel of the map $\mathbb{R}^3 \xrightarrow{A} \mathbb{R}$ given by the matrix $A = (1 \ \ -2 \ \ 2)$.

this matrix is already in RREF.

Variable x_1 is leading, variables x_2, x_3 are free.

We get a basis of ker A by setting a free variable to 1, other free variables to 0, and computing leading variables.

We do this for each free variable to get a basis.

$$x_2 = 1, \ x_3 = 0 \Rightarrow x_1 = 2x_1 - 2x_2 = 2 \qquad \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$x_2 = 0, \ x_3 = 1 \Rightarrow x_1 = -2 \qquad \vec{v}_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

The basis is $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

Section 4.1 #14. Consider set V of all sequences $(x_0, x_1, \ldots)$ that converge to 0, $\lim_{n\to\infty} x_n = 0$.

Is V a subspace?

(a) Does V contain the 0 vector of the ambient vector space of all sequences? that vector is the sequence $(0,0,0\ldots)$.

Yes, $(0,0\ldots,0\ldots)$ is in the subspace

(b) Is V closed under addition?

If $(x_0, x_1, \ldots)$ and $(y_0, y_1, \ldots)$ are in V, then their sum is

$(x_0+y_0, x_1+y_1, \ldots)$.

$$\lim_{n\to\infty}(x_n+y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = 0+0=0.$$

Yes, V is closed under addition

(c) V is closed under multiplication by scalars:

if $(x_0, x_1, \ldots) \in V$ and $k \in \mathbb{R}$, then

$$k(x_0, x_1, \ldots) = (kx_0, kx_1, \ldots)$$

$$\lim_{n\to\infty}(kx_n) = k \lim_{n\to\infty} x_n = k.0 = 0.$$

Therefore, all three conditions hold, and V is a linear subspace.

#28. Space of all matrices A that commute with $B=\begin{pmatrix}1&1\\0&1\end{pmatrix}$

$$A=\begin{pmatrix}a&b\\c&d\end{pmatrix} \qquad \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

$$\begin{pmatrix}a&a+b\\c&c+d\end{pmatrix}=\begin{pmatrix}a+c&b+d\\c&d\end{pmatrix}$$

We get a system of linear equations

$$\begin{cases}a=a+c\\a+b=b+d\\c=c\\c+d=d\end{cases} \iff \begin{cases}c=0\\d=a\\c=0\end{cases} \iff \begin{cases}c=0\\d=a\end{cases}$$

$$\Rightarrow A=\begin{pmatrix}a&b\\0&a\end{pmatrix}$$ we can take any a,b.

this set is a linear subspace. As a basis, we can take matrices of this form with $a=1, b=0$ and $a=0, b=1$.

$\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $\begin{pmatrix}0&1\\0&0\end{pmatrix}$ is a basis of the 2-dimensional space of matrices A that commute with B.

Section 4.2 #6. $T(M) = M \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ $T: \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $T(M) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} a+3b & 2a+6b \\ c+3d & 2c+6d \end{pmatrix}$ (or check $T(A+B) = T(A)+T(B)$)

All coefficients are linear in a,b,c,d, hence T is a linear map

$\ker T$ consists of matrices such that $T(M) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. this means

$$\begin{cases} a+3b=0 \\ 2a+6b=0 \\ c+3d=0 \\ 2c+6d=0 \end{cases} \quad \text{equations 2, 4 follow from equations 1, 3} \iff \begin{cases} a+3b=0 \\ c+3d=0 \end{cases} \Rightarrow \begin{matrix} a=-3b \\ c=-3d. \end{matrix}$$

We can take anything for b, d and they determine a, c.

Two free parameters $\Rightarrow \ker T$ has dimension 2

basis: $b=1, d=0 \Rightarrow a=-3, c=0 \Rightarrow \begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix}$

$b=0, d=1 \Rightarrow a=0, c=-3 \Rightarrow \begin{pmatrix} 0 & 0 \\ -3 & 1 \end{pmatrix}$

$\ker T$ has the basis $\begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -3 & 1 \end{pmatrix}$.

$\underbrace{\dim(\ker T)}_{=2} + \dim(\operatorname{im} T) = \dim \mathbb{R}^{2\times 2} = 4.$

$\Rightarrow \dim(\operatorname{im} T) = 2$ and $\operatorname{im} T$ is 2-dimensional.

To get a basis, we set free parameters b, d to 0 and leading variables $a=1, c=0$ and $a=0, c=1$, and apply T

$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$, $T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$.

basis of $\operatorname{im} T$ is $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$

T is not invertible since, for instance, it has a nontrivial kernel.