Section 3.2, #18

$$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \quad \vec{v}_4 \quad \vec{v}_5 \quad \vec{v}_6 \quad \vec{v}_7$$

$$\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}3\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}4\\5\\0\end{bmatrix}, \begin{bmatrix}6\\7\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$\vec{v}_1 = 0$, redundant (zero vector is always redundant)

$\vec{v}_3 = 3\vec{v}_2$, $\vec{v}_3$ is redundant

$\vec{v}_5 = 4\vec{v}_2 + 5\vec{v}_4$, $\vec{v}_5$ is redundant

$\vec{v}_6 = 6\vec{v}_2 + 7\vec{v}_4$, $\vec{v}_6$ is redundant

$\vec{v}_2$, $\vec{v}_4$, $\vec{v}_7$ are clearly <u>not</u> redundant

Redundant vectors: $\vec{v}_1$, $\vec{v}_3$, $\vec{v}_5$, $\vec{v}_6$.

#38 a) If the vectors $\vec{v}_1, \dots \vec{v}_m$ do not span V, choose a vector $\vec{v}_{m+1}$ in V but not in the span of $\vec{v}_1, \dots \vec{v}_m$.

Then $\vec{v}_1, \dots, \vec{v}_m, \vec{v}_{m+1}$ are linearly independent.

This contradicts the assumption that m is the largest number of linearly independent vectors we can find in V. Therefore, $\vec{v}_1, \dots \vec{v}_m$ is a basis of V.

b) By part (a) subspace V admits a basis $\vec{v}_1, \dots, \vec{v}_m$.

Each vector $\vec{v}_j$ can be represented via its column of coordinates

$$\vec{v}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$$

Put these columns together into a matrix

$$A = \left(a_{ij} \right) = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ | & | & & | \end{pmatrix}$$

Necessarily, $V = \operatorname{im}(A)$.

Section 3.3 #16.

$$A = \begin{pmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Matrix A is in RREF form. In an RREF matrix, redundant columns are exactly non-pivot columns.

In this example, columns 2 and 4 are redundant.

You can also see it directly, denoting columns $\vec{v_1}, \vec{v_2} .. \vec{v_5}$

$\vec{v_2} = -2\vec{v_1}$

$\vec{v_4} = -\vec{v_1} + 5 v_3$

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$$A=\begin{pmatrix}2&4&8\\4&5&1\\7&9&3\end{pmatrix}\begin{matrix}\times\frac{1}{2}\\ \\ \\ \end{matrix}\longrightarrow\begin{pmatrix}1&2&4\\4&5&1\\7&9&3\end{pmatrix}\begin{matrix}\\-4(\mathrm{I})\\-7(\mathrm{I})\end{matrix}\longrightarrow$$

$$\rightarrow\begin{pmatrix}1&2&4\\0&-3&-15\\0&-5&-25\end{pmatrix}\begin{matrix}\\ \times(-1/3)\\ \times(-1/5)\end{matrix}\longrightarrow\begin{pmatrix}1&2&4\\0&1&5\\0&1&5\end{pmatrix}\begin{matrix}\\ \\-(\mathrm{II})\end{matrix}\longrightarrow$$

$$\rightarrow\begin{pmatrix}1&2&4\\0&1&5\\0&0&0\end{pmatrix}\begin{matrix}-2(\mathrm{II})\\ \\ \\ \end{matrix}\longrightarrow\begin{pmatrix}1&0&-6\\0&1&5\\0&0&0\end{pmatrix}=\mathrm{rref}(A)$$

$\ker A=\ker(\mathrm{rref}(A))$. $\dim(\ker(A))$ is the # of free variables

There is one free variable x_3, and for $\vec{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ $\in \ker A$ we have

$x_1=6x_3$, $x_2=-5x_3$. Set $x_3=1 \Rightarrow x_1=6$, $x_2=-5$ and the

vector $\begin{pmatrix}6\\-5\\1\end{pmatrix}$ is a basis of $\ker A$.

Basis of $\mathrm{im}\,A$ is given by columns of A that correspond to leading columns of $\mathrm{rref}(A)$. In our example, these are columns 1 and 2.

Hence, basis of image of A is given by columns 1 and 2 of A,

$$\begin{pmatrix}2\\4\\7\end{pmatrix}, \begin{pmatrix}4\\5\\9\end{pmatrix}.$$

#26a)

$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. By inspection, we see that colums 2 and 3 are equal, and that's the only relation on columns of C. This means $\ker C$ is one-dimensional, with basis vector $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

In matrices H, T, X, Y last two columns are not equal. This means $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ is not in kernel subspaces for these matrices.

Excluding H, T, X, Y, we are left with L.

In L, last 2 columns are equal, and that's the only relation on columns of L. Therefore $\ker L = \ker C$, with basis vector $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.