Section 2.4, #4

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{matrix} -3(\mathrm{III}) \\ -2(\mathrm{III}) \end{matrix} \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) -2(\mathrm{II}) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Check

$$A \cdot A^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2+2 & 1-4+3 \\ 0 & 1 & -2+2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underline{I}.$$

Note that the inverse of an upper-triangular matrix is always upper-triangular.

An upper-triangular matrix is invertible if and only if all diagonal entries are non-zero (exercise).

Section 2.4 # 8 $\quad A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ -(\mathrm{I}) \\ -(\mathrm{I}) \end{array} \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ -2(\mathrm{II}) \end{array} \longrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \begin{array}{l} -(\mathrm{III}) \\ -2(\mathrm{III}) \\ \end{array} \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \begin{array}{l} -(\mathrm{II}) \\ \\ \end{array} \longrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

Check:

$$AA^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix} \begin{pmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 3-3+1 & -3+5-2 & 1-2+1 \\ 3-6+3 & -3+10-6 & 1-4+3 \\ 3-9+6 & -3+15-12 & 1-6+6 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

Section 2.4 #32

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad ad-bc=1 \quad , \quad A^{-1} = A$$

$$A^{-1} = \frac{1}{\det}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ since } \det(A) = ad-bc=1$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \Rightarrow \begin{matrix} a=d & b=-b \\ c=-c & d=a \end{matrix} \Rightarrow d=a, \; b=c=0$$

since also $ad-bc=1 \Rightarrow a \cdot a = 1 \Rightarrow a = \pm 1$.

2 solutions: 1) $a=d=1, b=c=0$ $\quad A=I$ identity matrix

2) $a=d=-1, b=c=0$ $\quad A=-I$ minus identity matrix.

Section 3.1, #16

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

im A is spanned by both columns, since they are not proportional

$$\operatorname{im}(A) = \operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right).$$

#30. For example, $A = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$, or $A = \begin{pmatrix} 1 & 2 \\ 5 & 10 \end{pmatrix}$.