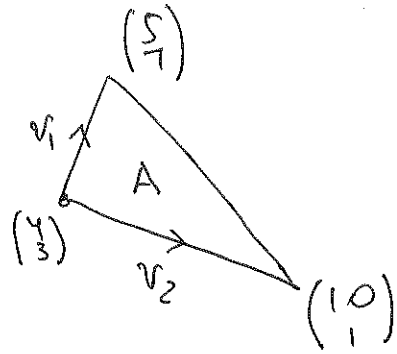
Homework #11

#3 Section 6.3



$$v_1 = \begin{pmatrix}5\\7\end{pmatrix} - \begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix}$$

$$v_2 = \begin{pmatrix}10\\1\end{pmatrix} - \begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}6\\-2\end{pmatrix}$$

$$\text{Area of triangle} = \frac{1}{2}\left|\det\begin{bmatrix}1 & 6\\ 4 & -2\end{bmatrix}\right| = \frac{1}{2}\left|-2-24\right| = 13$$

#4 $\quad v_1 = \begin{bmatrix}a_1 - b_1\\ a_2 - b_2\end{bmatrix} \qquad v_2 = \begin{bmatrix}a_1 - b_1\\ c_2 - b_2\end{bmatrix}$

$$\text{Area } A = \frac{1}{2}\left|\det\begin{pmatrix}a_1-b_1 & c_1-b_1\\ a_2-b_2 & c_2-b_2\end{pmatrix}\right| \qquad (*)$$

$$\det\begin{pmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ 1 & 1 & 1\end{pmatrix} = \det\begin{pmatrix}a_1-b_1 & b_1 & c_1-b_1\\ a_2-b_2 & b_2 & c_2-b_2\\ 0 & 1 & 0\end{pmatrix} = -\det\begin{pmatrix}a_1-b_1 & c_1-b_1\\ a_2-b_2 & c_2-b_2\end{pmatrix}.$$

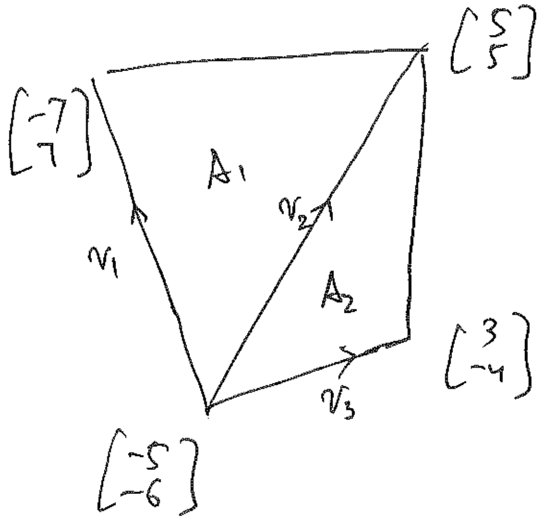
subtract column 2 from columns 1 & 3 does not change determinant

Laplace expansion along last column

Therefore

$$A = \frac{1}{2}\left|\det\begin{pmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ 1 & 1 & 1\end{pmatrix}\right|$$

#7



$$v_1 = \begin{bmatrix}-7\\7\end{bmatrix} - \begin{bmatrix}-5\\-6\end{bmatrix} = \begin{bmatrix}-2\\13\end{bmatrix}$$

$$v_2 = \begin{bmatrix}5\\5\end{bmatrix} - \begin{bmatrix}-5\\-6\end{bmatrix} = \begin{bmatrix}10\\11\end{bmatrix}$$

$$v_3 = \begin{bmatrix}3\\-4\end{bmatrix} - \begin{bmatrix}-5\\-6\end{bmatrix} = \begin{bmatrix}8\\2\end{bmatrix}$$

$$A_1 = \frac{1}{2}\left|\det\,[v_1\ v_2]\right| = \frac{1}{2}\left|\det\begin{bmatrix}-2 & 10\\13 & 11\end{bmatrix}\right| = \frac{1}{2}\left|-22-130\right| = \frac{1}{2}\cdot 152 = 76$$

$$A_2 = \frac{1}{2}\left|\det\,[v_2\ v_3]\right| = \frac{1}{2}\left|\det\begin{bmatrix}10 & 8\\11 & 2\end{bmatrix}\right| = \frac{1}{2}\left|20-88\right| = \frac{1}{2}\cdot 68 = 34$$

Area $A = A_1 + A_2 = 110$

#8 $A = [\vec{v}_1 \cdots \vec{v}_n]$ noninvertible, $\Rightarrow$ $\det A = 0$.

some $\vec{v}_i$ is redundant, a linear combination of $\vec{v}_1, \ldots \vec{v}_{i-1}$

$\Rightarrow \vec{v}_i^{\perp} = 0 \Rightarrow \quad |\vec{v}_1|\,|\vec{v}_2^{\perp}| \ldots |\vec{v}_n^{\perp}| = 0 = |\det A|$

#10) $A = \left| \vec{v}_1 \ldots \vec{v}_n \right|$

$| \det A | = |\vec{v}_1| \, |\vec{v}_2^{\perp}| \ldots |\vec{v}_n^{\perp}|.$

Note that for each $\vec{v}_i$, the length of $\vec{v}_i^{\perp}$ is less than or equal the length of $\vec{v}_i$:

$$|\vec{v}_i^{\perp}| \leq |\vec{v}_i|$$

Since $\vec{v}_i^{\perp}$ is the orthogonal projection of $\vec{v}_i$ onto $V^{\perp}$, where V is the span of $\vec{v}_1, \ldots \vec{v}_{i-1}$.

$\Rightarrow |\det A| = |\vec{v}_1| \, |\vec{v}_2^{\perp}| \ldots |\vec{v}_n^{\perp}| \leq |\vec{v}_1| \, |\vec{v}_2| \ldots |\vec{v}_n|.$

Equality holds $\left(|\det A| = |\vec{v}_1| \, |\vec{v}_2| \ldots |\vec{v}_n| \right)$ if and only if $\vec{v}_1, \vec{v}_2 \ldots \vec{v}_n$ are mutually orthogonal or one of them is 0.

#13

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$AA^T = \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix} \qquad \cancel{\det A} \quad \text{Area} = \sqrt{\det(A^T A)} = \sqrt{20}.$$

$$\det(AA^T) = 120 - 100 = 20$$

$$\boxed{\text{Area} = \sqrt{20}}$$

#22

$$\left| \begin{matrix} 3x + 7y = 1 \\ 4x + 11y = 3 \end{matrix} \right| \qquad A = \begin{bmatrix} 3 & 7 \\ 4 & 11 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\det A = 33 - 28 = 5 \neq 0.$$

$$A_{\vec{b},1} = \begin{bmatrix} 1 & 7 \\ 3 & 11 \end{bmatrix}$$

Cramer's rule: $\vec{x} = \dfrac{\det(A_{\vec{b},1})}{\det A} =$

$$\det A_{\vec{b},1} = -10$$

$$= \frac{-10}{5} = -2$$

$$A_{\vec{b},2} = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$$

$$\det A_{\vec{b},2} = 5$$

$$\vec{y} = \frac{\det(A_{\vec{b},2})}{\det A} = \frac{5}{5} = 1.$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$ unique solution