MIDTERM II SOLUTIONS

1a) $[0\ 0]\begin{bmatrix} x \\ y \end{bmatrix} = 0$ for all $x, y \in \mathbb{R}$ so

$\ker(A) = \mathbb{R}^2$, basis $= \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, $\dim(\ker) = 2$

b) $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow[R_1 \to R_1 - R_2]{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Relations: $x_1 = x_3$, $x_2 = -2x_3$

$\Rightarrow$ Kernel $= \left\{ s\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} : s \in \mathbb{R} \right\}$

Basis $= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $\dim(\ker) = 1$.

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2. $\vec{v}_2 = 0\vec{v}_1$; $\vec{v}_4 = 3\vec{v}_3$; $\vec{v}_5 = \vec{v}_1 + \vec{v}_3$

Basis of image: $\{\vec{v}_1, \vec{v}_3\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

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3. $S = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $S^{-1} = \dfrac{1}{3\cdot 1 - 2\cdot 2}\begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$

$$B = S^{-1}AS = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 5 & 8 \\ 11 & 18 \end{pmatrix} = \begin{pmatrix} 7 & 12 \\ -1 & -2 \end{pmatrix}$$

4a) $f(t) = a + bt + ct^2 \in V$ iff $f(1) = a + b + c = 0$

So $V = \{ a + bt + ct^2 \mid a, b, c \in \mathbb{R},\ a + b + c = 0 \}$

i) $0 = 0 + 0t + 0t^2 \in V$

ii) If $f(t) = a + bt + ct^2$ and $g(t) = p + qt + rt^2$ are in V, then $(f+g)(t) =$

$(a+p) + (b+q)t + (c+r)t^2$ satisfies

$(f+g)(1) = a+p + (b+q) + (c+r) =$
$(a+b+c) + (p+q+r) = 0$

iii) $kf(1) = k(a+b+c) = k \cdot 0 = 0$ if $k \in \mathbb{R}$, $f(t) = a + bt + ct^2 \in V$.

$\Rightarrow$ Linear subspace.

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b) $f(t) = a + bt + ct^2$, $f(1) = 0 \Rightarrow a = -b - c$

So $f(t) = -b - c + bt + ct^2 = -b(1-t) +$
$-c(1-t^2)$

Basis : $\{1-t,\ 1-t^2\}$. Dimension $= 2$

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5. $T(1) = 2 \cdot (1)' = 0 = 0(1) + 0 \cdot t + 0 \cdot t^2$

$T(t) = 2 \cdot (t)' = 2 = 2 \cdot (1) + 0 \cdot t + 0 \cdot t^2$

$T(t^2) = 2 \cdot (t^2)' = 4t = 0 \cdot (1) + 4 \cdot t + 0 \cdot t^2$

$\Rightarrow$ Matrix is $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

6. $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \|\vec{v}_1\| = \sqrt{1^2+1^2+1^2+1^2} = 2$

$\Rightarrow \quad \vec{u}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$

$\vec{V}_2 = \begin{pmatrix} 6 \\ 4 \\ 4 \\ 6 \end{pmatrix}, \quad \vec{V}_2^{\perp} = \vec{V}_2 - (\vec{V}_2 \cdot \vec{u}_1)\, \vec{u}_1$

$= \begin{pmatrix} 6 \\ 4 \\ 4 \\ 6 \end{pmatrix} - (3+2+2+3) \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$

$\vec{u}_2 = \dfrac{1}{\|\vec{v}_2^{\perp}\|} \vec{v}_2^{\perp} = \dfrac{1}{2} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}$

$\Rightarrow$ Orthonormal basis $\left\{ \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix} \right\}$

For "R" matrix, $r_{11} = \|\vec{v}_1\| = 2$

$r_{12} = \vec{u}_1 \cdot \vec{v}_2 = 10$

$r_{22} = \|\vec{v}_2^{\perp}\| = 2$

$r_{21} = 0$ (matrix is upper triangular)

$$\Rightarrow \begin{bmatrix} 1 & 6 \\ 1 & 4 \\ 1 & 4 \\ 1 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 2 & 10 \\ 0 & 2 \end{bmatrix}}_{R}$$

7. a) FALSE: $A: \mathbb{R}^4 \to \mathbb{R}^3$. Rank-Nullity

Says $\dim(\text{image}) + \dim(\ker) = 4$,

Rank is just $\dim(\text{image})$ which is 2, so

Nullity $= \dim(\ker) = 2$.

b) FALSE: If I_n and $-I_n$ are similar,

there exists some matrix S such that

$I_n = S^{-1}(-I_n)S$. But $-I_n S = S(-I_n)$

so RHS $\Rightarrow S^{-1}S(-I_n) = -I_n$

$\Rightarrow I_n = -I_n$, contradiction

c) TRUE: P_1 dimension 2, basis $1, t$

$\mathbb{C} = \{a+bi \mid a,b \in \mathbb{R}\}$ dimension 2

over $\mathbb{R}$, basis 1 i

Define $T: P_1 \to \mathbb{C}$ sending $1 \to 1$, $t \to i$

This is isomorphism.

d) TRUE: orthogonal matrices are square matrices

that satisfy $A^{-1} = A^T$. In

particular, they are invertible.

e) TRUE: General property of transposes.

Concretely, if $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$,

$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$ then $(A+B)^T = \begin{pmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \\ a_{13}+b_{13} & a_{23}+b_{23} \end{pmatrix}$

$= A^T + B^T$