Lie groups

Homework #2, due Monday, September29.

1. Let U be an open neighbourhood of the identity element 1 in a topological group G. Show that there is an open neighbourhood V of 1 with $V^{-1} = V$ and $V^2 \subset U$. Here $V^2 = \{v_1 v_2 | v_1, v_2 \in V\}$.

2. Use previous exercise to show that, if $\{1\}$ is a closed subgroup of a topological group G then G is hausdorff.

3. Show that any open subgroup H of a topological group G is also closed. (Hint: use coset decomposition).

4. Let U be a symmetric ($U^{-1} = U$) neighbourhood of 1 in a topological group G. Then $H = \bigcup_{n=1}^{\infty} U^n$ is an open (and closed) subgroup of G.

This implies that a connected topological group (in particular, a connected Lie group) is generated by any neighbourhood of 1.

5. Show that for any finitely-generated abelian group H there exists a Lie group G with $\pi_1(G) \cong H$.

6. Give an example of two matrices A, B such that $\exp(A + B) \neq \exp(A)\exp(B)$.

7. Show that the exponential map $\exp : Mat(n, \mathbb{C}) \longrightarrow GL(n, \mathbb{C})$ is surjective (hint: use Jordan normal form), but $\exp : Mat(n, \mathbb{R}) \longrightarrow GL(n, \mathbb{R})$ is not.

8. Determine the centers of Lie groups $O(n)$, $Sp(n)$, and $SL(n, \mathbb{R})$.