Lie groups

Homework #1, due Monday, September 21.

In the problems below, all representations are assumed to be over $\mathbb{C}$, unless specified otherwise.

1. Classify irreducible representations of the groups $\mathbb{Z}/n$, $\mathbb{Z}/2 \times \mathbb{Z}/3$, $S_3 \times \mathbb{Z}/2$.

2. Write down the character tables of the alternating group A_4, the symmetric group S_4, and the 8-element dihedral group D_4. Use the tables to find the dual of each irreducible reprentation of these groups.

3. Determine the structure constants of the representation rings $Rep(A_4)$ and $Rep(S_4)$ in the bases of irreducible representations (in other words, compute multiplicities of irreducible representations V_k in tensor products $V_i \otimes V_j$). Do not use brute force – utilize symmetries and properties of representations and use character tables from exercise 2.

4. (a) For each irreducible representation V of the symmetric group S_4 determine how $\Lambda^2 V$ and $S^2 V$ decompose into irreducibles.

(b) Do the same for the alternating group A_4.

5. Show that $\Lambda^2(V^*) \cong (\Lambda^2 V)^*$ for any representation V of a finite group.

5. (a) Give an example of a representation of $\mathbb{Z}$ which is not completely reducible.

(b) Show that $\Bbbk[G]$ is semisimple (every representation is completely reducible) if and only if G is finite and the characteristic p of $\Bbbk$ does not divide the order of G.

6. How to determine the kernel of a representation if you know its character?

7. Let $A = \mathbb{C}\langle x, \partial \rangle / (\partial x - x\partial - 1)$ be the ring of polynomial differential operators in one variable. Show that A has no finite-dimensional representations. Give an example of an infinite-dimensional irreducible representation of A.