Introduction to knot theory, Spring 2013

Homework 8, due Thursday, April 4

1. (10 points) Let C_1, C_2 be subcoalgebras of a coalgebra C. Show that $C_1 + C_2$ and $C_1 \cap C_2$ are subcoalgebras of C.

2. (10 points) (a) Define the notion of the product $C \times D$ of coalgebras C and D (the underlying vector space should be the direct sum $C \oplus D$).

(b) Take the group bialgebra $F[G]$ and consider it as coalgebra only (forget multiplication, and $\Delta(g) = g \otimes g$ for $g \in G$). How can you decompose it into a direct product of coalgebras?

3. (10 points) We say that a linear subspace I of a bialgebra H is an ideal if $HI \subset I, IH \subset I$ and $\Delta(I) \subset H \otimes I + I \otimes H$, $\epsilon(I) = 0$. Prove that the quotient vector space H/I has a natural bialgebra structure inherited from H.

4. (15 points) Let F be a field of finite characteristic p and $H = F[x]$ the polynomial algebra on one generator. Make H a bialgebra by setting $\Delta(x) = x \otimes 1 + 1 \otimes x$ (so that x is primitive) and $\epsilon(x^n) = \delta_{0,n}$. Check that

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}.$$

Use this to show

$$\Delta(x^p) = x^p \otimes 1 + 1 \otimes x^p$$

given our assumption on F. Conclude that x^p is primitive. Let I be the algebra ideal of $F[x]$ generated by x^p. Check that I is also a bialgebra ideal (see exercise 3). Conclude that $F[x]/(x^p)$ is a bialgebra.

5. (15 points for (a)-(c)) Let A be an algebra over a field F. A derivation $D : A \longrightarrow A$ is a linear map such that $D(ab) = D(a)b + aD(b)$.

(a) Check that derivations constitute a vector space over F.

(b) Chat that the commutator of derivations is a derivation, where $[D_1, D_2]$ is defined by $[D_1, D_2](x) = D_1 D_2(x) - D_2 D_1(x)$, $x \in A$.

(c) Given $a \in A$, define D_a to be the endomorphism of A which takes $b \in A$ to $D_a(b) = ab - ba$. Show that D_a is a derivation. Such derivations are called inner.

(d) (*extra credit*) Prove that any derivation of the matrix algebra $Mat(n, F)$ is inner.

6. (10 points) Given an associative algebra A, define a bracket on it by $[x, y] = xy - yx$. Check that A equipped with the bracket satisfies the Jacobi identity. We denote the resulting Lie algebra by A^L or A^{Lie}.