Introduction to knot theory, Spring 2012

Homework 2, due Thursday, February 7

1. (10 points) The mirror image $K^!$ of a knot K is given by reflecting K about a plane in $\mathbb{R}^3$. Let D be a diagram of K, D_1 be the diagram obtained from D by inverting all crossings, and D_2 - diagram given by reflecting D about a line in the plane. Show that both D_1 and D_2 are diagrams of $K^!$. Use this to prove that coloring groups $C(K)$ and $C(K^!)$ are naturally isomorphic. Conclude that $\tau_n(K) = \tau_n(K^!)$ for any n.

2. (20 points) Compute the coloring groups $C(K)$ for the trefoil 3_1, figure-eight knot 4_1, and the five-crossing knot 5_2 (get diagrams for these knots from the Rolfsen knot table). Using the coloring groups, determine $\tau_p(3_1)$ and $\tau_p(4_1)$ for all primes p.

3. (10 points) Compute the coloring group of the Borromean rings and determine the number of its p-colorings for all primes p.

4. (10 points) Suppose that a knot K is the closure of braid σ. Explain how to construct a braid whose closure is $K^!$.

5. (20 points) (a) Draw the closure of the 3-stranded braid $\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1$. Check that the closure is a 2-component link and compute the linking number of the two components.
(b) Given a 3-stranded braid σ, how can you quickly tell if its closure is a link or a knot? Try your method on the following braids:

$$\sigma_1^{41}\sigma_2^{-73}, \qquad (\sigma_1^2\sigma_2)^{1000}, \qquad (\sigma_1\sigma_2^{-1}\sigma_1)^{51}, \qquad (\sigma_2\sigma_1)^{211}.$$

In each case, determine whether the closure is a knot, a 2-component link, or a 3-component link.

6. (10 points) Give an example of a braid whose closure is the figure-eight knot.

Extra credit:

I. Give an example of 3-stranded braids σ and τ such that each of the closures $\widehat{\sigma}$, $\widehat{\tau}$ and $\widehat{\sigma\tau}$ is the unknot. Next, determine whether the closure of the braid $\sigma^{-1}\tau$ is the unknot.

II. Show that the number of 2-colorings is a trivial invariant of knots.