Algebraic topology, Fall 2014

Homework 5, due Monday, October 20

1. How would you define a $K(G, 0)$ space, where G is a group?

2. In this exercise we assume that A, B, C are complexes of abelian groups, although everything works for complexes of modules over any ring R. We denote by $\mathrm{Kom}(\mathbb{Z})$ the category of complexes of abelian groups and their morphisms.

(a) Given a morphism $f : A \longrightarrow B$ of complexes, explain how to define complexes $\ker(f)$ and $\mathrm{coker}(f)$ (the kernel and cokernel of f).

Given two complexes A and B, a homotopy h is a collection of homomorphisms $h_n : A_n \longrightarrow B_{n+1}$. We say that $f : A \longrightarrow B$ is null-homotopic if $f = dh + hd$ for some h.

(b) Check that f, for any homotopy h, is a homomorphism of complexes.

(c) Any homomorphism of complexes induces a homomorphism of their homology groups. Show that a null-homotopic morphism induces the zero map $H(A) \longrightarrow H(B)$ and, more generally, homotopic morphisms induce the same map on homology.

(d) Given three complexes and two homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ the composition gf is null-homotopic if either f or g is null-homotopic.

(e) The sum and difference of null-homotopic maps is null homotopic.

Conclude that null-homotopic maps constitute an ideal in the category of complexes. The quotient by this ideal is called the homotopy category of complexes, denoted $\mathrm{Com}(\mathbb{Z})$.

3. We say that a complex A is contractible if and only if the identity map 1_A is null-homotopic. Show that A is contractible iff it is isomorphic in $\mathrm{Com}(\mathbb{Z})$ to the zero complex. Explain why the complex of abelian groups

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0,$$

with the obvious differential, is acyclic (has zero homology groups) but not contractible.

4. Using the classification of complexes of vector spaces obtained in

class, show that a complex of vector spaces of acyclic if and only if it is contractible.

Hatcher exercises 1, 4, 5, 15, 16, 17 in Section 2.1 pages 131-132.