Representations of finite groups, spring 2016
Homework 9, due Wednesday April 6 before class.

Topic: Tensor products of modules, induction and restriction of representations.

1. (10 points) Think through and prove the distributivity property of the tensor product

$$(M_1 \oplus M_2) \otimes_R N \;\cong\; (M_1 \otimes_R N) \oplus (M_2 \otimes_R N).$$

2. (20 points) (a) Using the formula $R/I \otimes R/J \cong R/(I+J)$ for a commutative ring R and ideals I, J, compute the following tensor products of $\mathbb{Z}$-modules:

$$\mathbb{Z}/10 \otimes \mathbb{Z}/5, \quad \mathbb{Z}/4 \otimes \mathbb{Z}/6, \quad \mathbb{Z}/4 \otimes (\mathbb{Z} \oplus \mathbb{Z}/2), \qquad (\mathbb{Z}/10 \oplus \mathbb{Z}) \otimes (\mathbb{Z}/5 \oplus \mathbb{Z}/9).$$

(In class we discussed but did not prove this formula. It's good to spend some time thinking about it and working out a couple of examples on your own, for $R = \mathbb{Z}$ or $R = F[x]$ and principal ideals generated by small numbers or polynomials of small degree. Even better if you prove the formula on your own.)

(b) Compute the following tensor products of $\mathbb{C}[x]$-modules:

$$\mathbb{C}[x]/(x) \;\otimes\; \mathbb{C}[x]/(x-1), \qquad \mathbb{C}[x]/(x^2) \;\otimes\; \mathbb{C}[x]/(x^3),$$
$$\mathbb{C}[x]/(x-1) \otimes \mathbb{C}[x]/(x^2+1).$$

3. (10 points) Take the dihedral group D_4 and its rotation subgroup C_4. Denote irreps of C_4 by $W_0, ..., W_3$ and irreps of D_4 by $V_0, \dots, V_4$. Write down the character tables of D_4 and C_4. Compute the characters of induced representations $W_i \uparrow D_4$ (hint: use normality of C_4) and then determine how these induced representations decompose into irreducibles $V_0, \dots, V_4$.

4. (20 points) Consider the inclusion of groups $S_3 \subset S_4$. Denote irreducible reps of S_3 by W_0, W_1, W_2 (trivial, sign, 2-dimensional), and irreps of S_4 by $V_0, V_1, \dots, V_4$. Recall and write down the character tables of S_3 and S_4. For each W_i, compute the character of the induced representation $W_i \uparrow S_4$, using the character formula derived in class. Determine multiplicities of irreducibles V_j in these induced representations. One way to check for consistency is by verifying that your formulas give the regular representation of S_4 if you start with the regular representation of S_3, which is the direct sum of W_i's with multiplicities given by their dimensions.