Representations of finite groups, spring 2016
Homework 11, due Wednesday April 27 before class.

1. Read the proof of the Spectral Theorem for Hermitian matrices (Steinberg, p.9, 2.3.6). Rewrite the proof to show that a unitary matrix A has eigenvalues of absolute value 1 and can be conjugated by a unitary matrix B to a diagonal matrix BAB^{-1}.

2. (a) Show that an element of $SU(2)$ has trace 0 iff it has order 4.

(b) Show that all elements of order 3 in $SU(2)$ are conjugate. Are all elements of order 5 conjugate in $SU(2)$; why?

3. Can you give an example of a finite subgroup H of $U(2)$ so that the fundamental representation $V = \mathbb{C}^2$ of $U(2)$, restricted to H, is not self-dual? In your example, is V irreducible?

4. Consider the affine Dynkin graph $\widetilde{D}_{n+2}$ with $n+3$ vertices corresponding to the binary dihedral group D_n^*. Looking at this graph, what can you say about the abelian groups $H = D_n^*/[D_n^*, D_n^*]$ and $H' = D_n/[D_n, D_n]$? Which nodes of the graph correspond to irreducible representations of D_n? Describe automorphisms of the graph induced by tensoring irreducible representations of D_n^* with various one-dimensional representations of D_n^*.

5. Prove the following **Theorem:** *If V is a complex faithful representation of a finite group G and W an irreducible representation of G, then W is a subrepresentation of $V^{\otimes n}$ for some n.* Hint: read the proof in the notes for the case G is a subgroup of $SU(2)$ and V the fundamental representation. In general, consider the intersection H of G with the center of $GL(V)$. Show that elements of H are the only ones with absolute value of character at least $d = \dim(V)$ and investigate

$$\frac{1}{d^n}(\chi_W, \chi_V^n)$$

in the limit of large n.