Representations of finite groups, spring 2016

Homework 1, due Wednesday January 27 before class.

1. From the axioms, deduce that $0 \in R$ acts trivially on any R-module M, that is $0m = 0$ for any $m \in M$.

2. (a) Let V be a 2-dimensional vector space over a field F with basis $\{v_1, v_2\}$. Consider it as a module over $F[x]$, with x acting via the linear operator X that takes v_1 to 0 and v_2 to v_1. Find all submodules of the $F[x]$-module V. Which of these submodules are proper?
(b) Find all submodules of abelian groups $\mathbb{Z}/10$ and $\mathbb{Z}/2 \times \mathbb{Z}/2$, viewed as modules over the ring $\mathbb{Z}$.

3. Given an R-module M and a submodule $N \subset M$, check that the quotient abelian group M/N is naturally an R-module via the action $r(m + N) = rm + N$. (You'll need to check, among other things, that this action is well-defined and does not depend on the choice of a coset representative.)

4. Given an R-module N and submodules $M_1, M_2 \subset N$, show that $M_1 + M_2$ (internal sum, defined in class) is a submodule of N. Under what conditions is the natural homomorphism from $M_1 \oplus M_2$ to $M_1 + M_2$ an isomorphism?

5. Which of the following rings are naturally $\mathbb{Q}$-algebras, where $\mathbb{Q}$ is the field of rational numbers?

$$\mathbb{Z},\ \mathbb{R},\ \mathbb{F}_9,\ \mathbb{Q}[x, x^{-1}],\ \mathbb{C}[x, y],\ \mathbb{Z}/20.$$

6. (a) Recall the notion of a monoid. Give your definition of what it means for a monoid G to act on a set X. Keep in mind the analogy with group actions. How would you define orbits of an action of G on a set?
(b) (optional) Consider the elevator monoid $E = \{1, e \mid e^2 = e\}$. Classify possible types of action of E on sets. How does your classification change if instead you use monoid $E' = \{1, e \mid e^3 = e\}$ with three elements $\{1, e, e^2\}$?