Name:

Representations of finite groups. Midterm exam.

In all problems below only complex representations are considered.

1. (20 points) Mark those squares that are followed by correct statements.

☐ The composition of intertwiners is an intertwiner.

☐ Any one-dimensional representation of a simple group is trivial.

☐ Any intertwiner from a representation V to itself is a multiple of the identity.

☐ Any representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is one-dimensional.

☐ If $\langle \chi_V, \chi_V \rangle > 1$ then V is reducible.

☐ There exist representations V and W such that $\langle \chi_V, \chi_W \rangle = -1$.

☐ The inner product of characters is symmetric, $\langle \chi_V, \chi_W \rangle = \langle \chi_W, \chi_V \rangle$.

☐ Any irreducible representation of S_4 is faithful.

☐ There exists a group with exactly 2006 isomorphism classes of irreducible representations.

2. (10 points) Determine the character table of the alternating group A_4. What are the multiplicities of irreducible representations of A_4 in the regular representation?

3. (10 points) (a) Give the definition of an action of a group G on a set X.

(b) Give the definition of an intertwiner $\alpha : V \to W$ between representations of G.

4. (10 points) Among the groups below, select those that have exactly three isomorphism classes of irreducible representations:

$$\mathbb{Z}_5, \qquad \mathbb{Z}_3, \qquad S_4, \qquad A_5, \qquad S_3.$$

For selected groups, list dimensions of irreducible representations.

5. (10 points) Give an example of a group G and representations V and W such that $\dim \operatorname{Hom}_G(V, W) = 3$.

Extra credit: Show that the dimension of an irreducible representation of G is at most $\sqrt{|G|}$. Find all groups G which have an irreducible representation of this dimension.