Midterm exam, Wednesday November 4.

NAME:

Books, notebooks, laptops, calculators, etc. are not allowed on the exam. All representations are considered over complex numbers.

1 (15 points). Mark the squares that are followed by correct statements.

☐ Any free module over the ring $\mathbb{R}[x]$ is irreducible.

☐ The set of even integers is an ideal of $\mathbb{Z}$.

☐ The sum $I + J$ of two ideals of a commutative ring R is an ideal of R.

☐ Any module over $\mathbb{Z}$ is completely reducible.

☐ The sum of two idempotents in a commutative ring R is an idempotent.

2 (15 points). Mark the squares that are followed by correct statements.

☐ The group $\mathbb{Z}/3 \times \mathbb{Z}/3$ has 6 isomorphism classes of irreducible representations.

☐ Any irreducible representation of a finite group G is contained in the regular representation $\mathbb{C}[G]$.

☐ Any one-dimensional representation of the group $\mathbb{Z}/5$ is irreducible.

☐ The group $\mathbb{Z}$ has infinitely many isomorphism classes of irreducible representations.

3 (10 points). Among the rings below, circle those that are commutative.

$$\mathbb{Z} \qquad \mathbb{Z} \times \mathbb{R} \qquad \mathbb{H} \qquad \mathrm{Mat}_2(\mathbb{C}) \qquad \mathbb{Z}/5 \qquad \mathbb{C}[\mathbb{Z}/4] \qquad \mathbb{R}[S_3]$$

Here $\mathbb{R}[S_3]$ is the group algebra of S_3 with real coefficients, and $\mathrm{Mat}_2(\mathbb{C})$ is the algebra of 2 by 2 matrices with complex coefficients.

4 (20 points) (a) Write down the character table of the group $\mathbb{Z}/3$.

(b) What's the character of the regular representation $V = \mathbb{C}[\mathbb{Z}/3]$ of $\mathbb{Z}/3$?

(c) What are the multiplicities of irreducible representations in $V \otimes V$?

5 (40 points). Let V_0, V_1, V_2 be the trivial, the sign, and the two-dimensional irreducible representations of the symmetric group S_3.

(a) Write down the character table of S_3.

(b) Find the characters and dimensions of the following representations:

$$S^2(V_2), \qquad \Lambda^2(V_1), \qquad \Lambda^2(V_0 \oplus V_1).$$

Which of these representations are irreducible?

(c) Among the following representations, circle those that are irreducible:

$$V_0 \otimes V_2 \qquad V_1 \otimes V_2 \qquad V_2 \otimes V_2 \qquad \Lambda^2(V_2), \qquad S^2(V_1).$$

(d) Which irreducible representations of S_3 are self-dual? Which are faithful?

(e) Find multiplicities of irreducible representations in $V_2 \otimes V_2 \otimes V_2$.

6^* **(optional)** Suppose that representation V of a group G is reducible (i.e. not irreducible). Prove that $S^2(V)$ is reducible as well. Give an example when $\Lambda^2(V)$ is irreducible.