Representations of finite groups

Homework #9, due Wednesday, November 18.

1. (a) Give an example of a category with an initial object but no terminal object.

(b) Give an example of a category with a terminal object but no initial object.

2. (a) Compute the following tensor products of $\mathbb{Z}$-modules:

$$\mathbb{Z} \otimes (\mathbb{Z} \oplus \mathbb{Z}/2), \qquad (\mathbb{Z}/10 \oplus \mathbb{Z}/3) \otimes (\mathbb{Z}/5 \oplus \mathbb{Z}/9).$$

(b) Compute the following tensor products of $\mathbb{C}[x]$-modules:

$$\mathbb{C}[x]/(x) \otimes \mathbb{C}[x]/(x+1), \qquad \mathbb{C}[x]/(x^2) \otimes \mathbb{C}[x]/(x^3),$$
$$\mathbb{C}[x]/(x-1) \otimes \mathbb{C}[x]/(x^2+1).$$

3. (a) Explain how to determine the character of the restricted representation $\mathrm{Res}(V)$ of $H \subset G$ given that V has character χ.

(b) Using characters, show that the restriction of the regular representation $\mathbb{C}[G]$ of G to H is isomorphic to the direct sum of $[G:H]$ copies of the regular representation $\mathbb{C}[H]$ of H. Here G and H are finite groups.

4. (a) Show that the trivial representation of G restricts to the trivial representation of its subgroup H.

(b) What happens when you restrict the sign representation of the symmetric group S_n to its subgroup S_{n-1} which consists of permutations that fix n?

(c) Take the fundamental representation $\mathbb{C}_4^0$ of S_4. Determine how it decomposes into irreducibles upon restriction to $S_3 \subset S_4$.