Representations of finite groups

Homework #6, due Wednesday, October 21.

1. (a) Show that a matrix A is unitary if and only if A^T is unitary.

(b) A is unitary iff $\overline{A}$ is unitary.

(c) Show that the determinant of a unitary matrix is a unit complex number (complex number of unit length).

(d) If A is unitary and diagonal, what can you say about its coefficients? Describe the group of unitary 1×1 matrices.

2. Give an example of a representation V of the infinite cyclic group $\mathbb{Z}$ such that $\chi_V(g) = \frac{1}{2}$, where g is a generator of $\mathbb{Z}$.

3. (a) Representation W of the symmetric group S_3 has character

$$\chi_W(1) = 7, \qquad \chi_W((12)) = -1, \qquad \chi_W((123)) = 4.$$

Find multiplicities of irreducible representations of S_3 in W and its dimension.

(b) Does there exist a representation of S_3 with the character

$$\chi_V(1) = 2, \qquad \chi_V((12)) = 2, \qquad \chi_V((123)) = -1?$$

4. In class we wrote down a presentation for D_8 (the group of symmetries of a square) with generators g, r and defining relations

$$g^4 = 1, \qquad r^2 = 1, \qquad rgr = g^{-1}.$$

(a) List conjugacy classes of D_8 and determine the number of irreducible representations.

(b) Find the commutator subgroup $[D_8, D_8]$ and the quotient $D_8/[D_8, D_9]$ by the commutator. Classify one-dimensional representations of D_8.

(c) Find the remaining (not one-dimensional) irreducible representations of D_8 and compute its character table.

5* (optional) Show that $\mathbb{C}_0^n$ is irreducible as a representation of the alternating group A_n if $n \geq 4$ (we showed in class that $\mathbb{C}_0^n$ is an irreducible representation of S_n for all n and an irreducible representation of A_n when $n = 4$).