Representations of finite groups

Homework #3, due Wednesday, September 30.

1. Which of the following are ideals of the ring $R = \mathbb{C}[x, y]$:
1) $I = \{0\}$,
2) $I = \mathbb{C}[x, y]$,
3) $I = \{f(x, y) \in R | f(1, -3) = 0\}$,
4) $I = \{f(x, y) \in R | f(1, -1) = 2\}$,
5) $I = \{f(x, y) \in R | f(0, 0) = f(5, 6)\}$?

2. (a) Simplify the following element of the group algebra $\mathbb{C}[\mathbb{Z}/3]$ of the cyclic group $\mathbb{Z}/3 = \{1, x, x^2\}, x^3 = 1$:

$$(2 - 3x + x^2)(1 + 2x + x^3 + x^4).$$

(b) Simplify the following element of the group ring $\mathbb{Z}[S_3]$ of the symmetric group S_3:

$$(1 + 2(13) - (132))((12) + (123) + 3(23)).$$

3. Give an example of a $\mathbb{Z}[x]$-module which is not completely reducible.

4. (a) Show that any group homomorphism $\alpha : G \longrightarrow H$ induces a homomorphism of group algebras $F[G] \longrightarrow F[H]$ (describe the latter homomorphism in terms of α).
(b) The map $g \longmapsto g^{-1}$ extends by linearity to an operator $F[G] \longrightarrow F[G]$ taking $\sum a_g g$ to $\sum a_g g^{-1}$. Show that this operator is a homomorphism of rings if and only if G is abelian.

5. (a) Show that a subset I of a group algebra $F[G]$ is a left ideal iff it's an F-vector space which is closed under left multiplications by elements of G.
(b) Let $I \subset F[G]$ consists of sums $\sum a_g g$ such that $\sum a_g = 0$. Show that I is a left ideal of $F[G]$.
(c) Suppose that G is finite. Then $J = F \sum_{g \in G} g$ is also a left ideal of $F[G]$.
(d) Check that the I and J in (b) and (c) above are 2-sided ideals.

6*(optional). Classify all ideals of the group algebra $\mathbb{C}[\mathbb{Z}/3]$.