Representations of finite groups

Homework #10, due Wednesday, November 25.

1. Write down irreducible representations of the groups $\mathbb{Z}/2$ and $\mathbb{Z}/4$ and determine what happens to these representations under induction and restriction functors corresponding to the inclusion $\mathbb{Z}/2 \subset \mathbb{Z}/4$ (hint: use multiplicative notation for these groups). Compute the Frobenius matrix of the restriction functor. What's the matrix of the induction functor?

2. Consider the inclusion of symmetric groups $S_3 \subset S_4$. Recall character tables of S_3 and S_4 that we obtained in class. Label irreps of S_3 and S_4 by V_i's and W_j's respectively. Compute the character of $\mathrm{Res}(W_j)$ for each j and find out how these restricted representations decompose into irreducible representations V_i's of S_3. Determine the matrix of the restriction functor.

3. The alternating group A_3 (isomorphic to the cyclic group $\mathbb{Z}/3$) is a subgroup of S_3. For each irreducible representation V_i, $i = 0, 1, 2$ of A_3 compute the character of the induced representation $\mathrm{Ind}(V_i)$ of S_3 and use it to decompose $\mathrm{Ind}(V_i)$ into irreducibles. Write down the matrix of the induction functor. What is the matrix of the adjoint restriction functor?

4. Let $Sets$ be the category of sets, Ab the category of abelian groups, and Gr the category of groups.
(a) Show that the forgetful functor $G_1 : Gr \longrightarrow Sets$ has a left adjoint functor F_1 that to a set X associates the free group on this set, $F_1(X) = \mathbb{Z}^{*X}$.

(b) Show that the forgetful functor $G_2 : Ab \longrightarrow Sets$ has a left adjoint functor F_2 that to a set X associates the free abelian group on this set, $F_1(X) = \mathbb{Z}^X$.

5. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$ be functors. Give a definition of the composition $G \circ F : \mathcal{A} \longrightarrow \mathcal{C}$ and check that it is a functor.

6. Give an example of a functor from the category of topological spaces to the category of real vector spaces.