Final exam, Monday Dec 21.

NAME:

Books, notebooks, laptops, calculators, etc. are not allowed on the exam. All representations are considered over complex numbers.

1 (15 points). Mark the squares that are followed by correct statements.

☐ The ring $\mathbb{Z}$ is commutative.

☐ Any ideal in a commutative ring is principal.

☐ The product of two idempotents in a commutative ring is an idempotent.

☐ Any division ring is commutative.

☐ Direct product $R_1 \times R_2$ of division rings is a division ring.

2 (20 points). (a) Recall that a ring A is called *semisimple* if every module over A is completely reducible. Among the rings below, circle those that are semisimple

$$\mathbb{R} \qquad \mathbb{C}[\mathbb{Z}/3] \qquad \mathbb{Q} \qquad \mathbb{R}[x] \qquad \mathbb{R}[S_4] \qquad \mathrm{Mat}(2, \mathbb{C}) \qquad \mathrm{Mat}(2, \mathbb{Q})$$

(b) Either prove the following statement or provide a counterexample:
Any subring of a semisimple ring is semisimple.

3. (10 points). Find all idempotents in the ring $\mathbb{Z}/20\mathbb{Z}$. What does existence of these idempotents tell you about the structure of the ring?

4 (15 points). Mark the squares that are followed by correct statements.

☐ The character of V is multiplicative: $\chi_V(gh) = \chi_V(g)\chi_V(h)$ for all $g, h \in G$ if and only if V is one-dimensional.

☐ The quaternion group Q_8 has four isomorphism classes of irreducible representations.

☐ Any irreducible representation of Q_8 is self-dual.

☐ The character of the direct sum is the direct sum of characters:
$\chi_{V \oplus W} = \chi_V + \chi_W$.

☐ Dihedral group D_{20} has an irreducible representation of dimension three.

5 (50 points). Let V_0, V_1, V_2 be the trivial, the sign, and the two-dimensional irreducible representations of the symmetric group S_3.

(a) Find the characters and dimensions of the following representations:

$$S^2(V_0), \qquad \Lambda^2(V_1), \qquad \Lambda^2(V_2), \qquad S^2(V_0 \oplus V_1).$$

Which of these representations are irreducible?

(b) Among the following representations, circle those that are irreducible:

$$V_0 \otimes V_0 \qquad V_2 \otimes V_2 \qquad V_1 \otimes V_2 \otimes V_1 \qquad S^2(V_2) \qquad S^2(V_1 \otimes V_1)$$

(c) Find multiplicities of irreducible representations in $V_2 \otimes V_2 \otimes V_2 \otimes V_1$.

(d) Give an example of a representation V of S_3 such that $\dim(V) = 5$, $\chi_V((12)) = -1$, and $\chi_V((132)) = 2$.

(e) Representation W of S_3 has the property $\langle \chi_W, \chi_W \rangle = 3$. Can you determine W?

(f) Representation T of S_3 satisfies $\chi_T \chi_T = \chi_T$. Can you determine T?

6 (30 points). (a) Write down the character tables of group A_3 and S_3.

(b) Inclusion of groups $A_3 \subset S_3$ induces induction and restriction functors. Find multiplicities of irreducible representations of A_3 in the restrictions of irreducible representations of S_3 to A_3. Also find multiplicities of irreps of S_3 in representations induced from irreps of A_3.

7 (30 points) (a) If any irreducible representation of a finite group G is one-dimensional, then G is abelian. Prove this claim.

(b) Suppose that a finite group G has only two isomorphism classes of irreducible representation. Show that G is isomorphic to $\mathbb{Z}/2$.

8 (40 points). List all finite subgroups of $SU(2)$. For each subgroup G draw its affine Dynkin graph. Write down the numbers d_i assigned to vertices. For each G determine the number of vertices in its Dynkin graph as well as the number of one-dimensional representations of G. Which of these groups G admit an irreducible representation of dimension 4?

9 (optional) Give an example of an irrep V of a finite group G such that $\Lambda^2(V)$ has a proper subrepresentation (i.e., not irreducible).