§7. Biadjoint Functors

Biadjoint functors and string diagrams

First, we recall the notion of adjoint functors.

A pair of functors (F, G) between two categories $\mathcal{A}$ and $\mathcal{B}$:

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] & \mathcal{B} \arrow[l, bend left, "G"] \end{tikzcd}$$

is said to be an adjoint pair if $\forall\, M \in Ob(\mathcal{A}),\ N \in Ob(\mathcal{B})$,

$$\mathrm{Hom}_{\mathcal{B}}(FM, N) \cong \mathrm{Hom}_{\mathcal{A}}(M, GN),$$

and this isomorphism is functorial in M and N.

If we set $N = FM$, we have an isomorphism:

$$\mathrm{Hom}_{\mathcal{B}}(FM, FM) \cong \mathrm{Hom}_{\mathcal{A}}(M, GFM),$$

functorial in M. The image of id_{FM} picks out a distinguished element in $\mathrm{Hom}_{\mathcal{A}}(M, GFM)$. In other words, there is a 2-functor (i.e. natural transformation of functors):

$$\alpha : \mathrm{Id}_{\mathcal{A}} \Longrightarrow GF.$$

Similarly, if we set $M = GN$, we would get

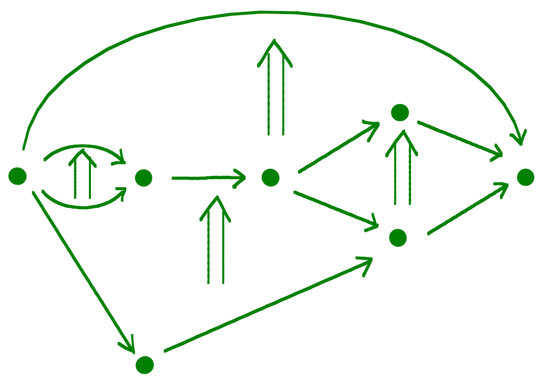
$$\beta : FG \Longrightarrow \mathrm{Id}_{\mathcal{B}}.$$

It's readily checked that these 2-functors satisfy the coherence relation:

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\mathrm{Id}_F \circ \alpha"] \arrow[rr, bend right, "\mathrm{Id}_F"'] & FGF \arrow[r, Rightarrow, "\beta \circ \mathrm{Id}_F"] & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, Rightarrow, "\alpha \circ \mathrm{Id}_G"] \arrow[rr, bend right, "\mathrm{Id}_G"'] & GFG \arrow[r, Rightarrow, "\mathrm{Id}_G \circ \beta"] & G \end{tikzcd}$$

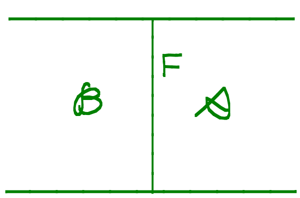
Def. We say that (F, G) is an adjoint pair if there exist 2-functors α, β s.t. the above coherence condition holds.

We now turn to the diagrammatics of 2-categories and adjoint functors. In the traditional notation, functors between categories, natural transformations between functors are usually depicted as:

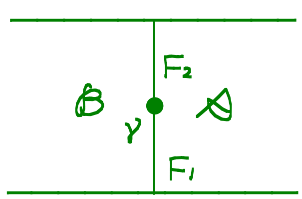


Instead of using this diagrammatics, we will introduce string diagrams, which are planar diagrams Poincaré dual to the traditional depiction.

We label categories by regions, functors by lines separating regions, and 2-functors by labels dividing lines: (read from bottom to top, right to left):

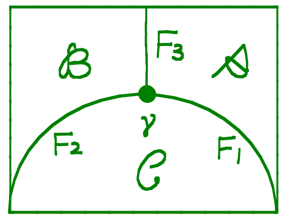


: a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$



: a 2-functor $\gamma : F_1 \Longrightarrow F_2$

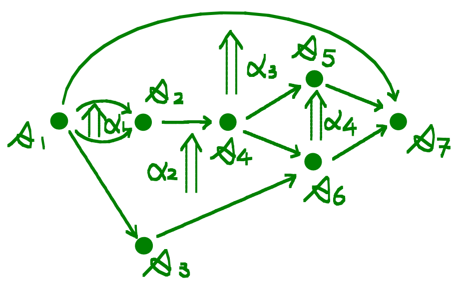
Then



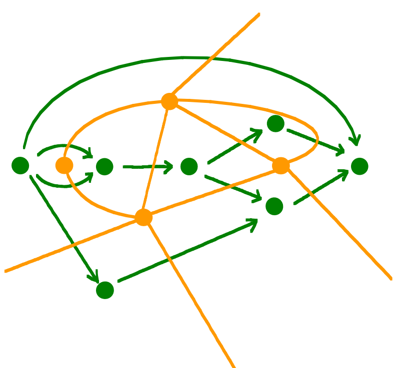
depicts a natural transformation:

$$\gamma : F_2 \circ F_1 \Longrightarrow F_3$$

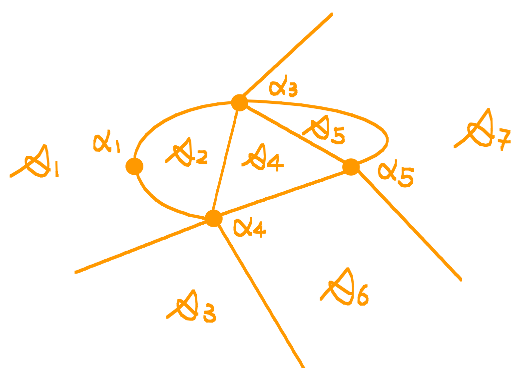
If we use string diagrams to depict the classical diagram above,



we get

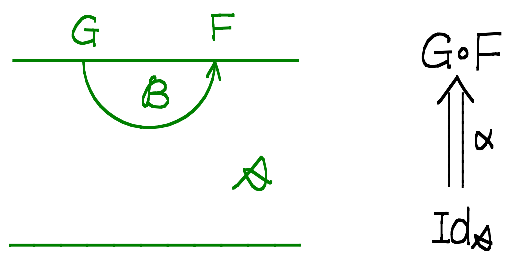


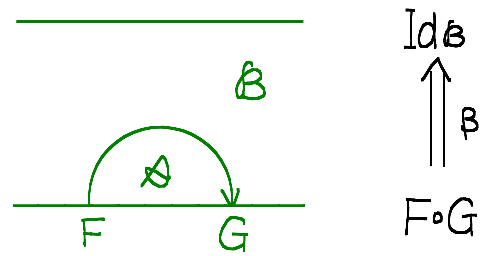
Poincaré dual



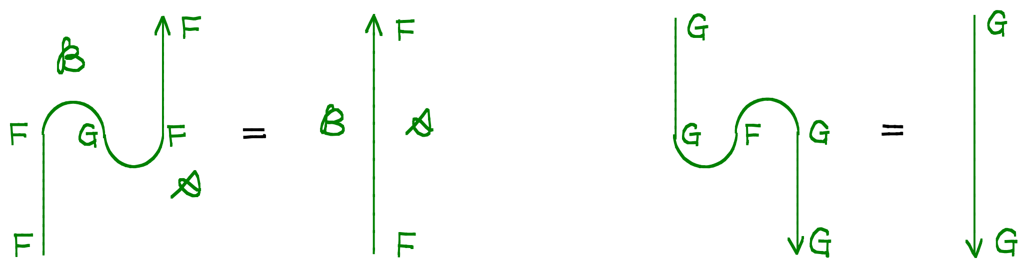
String diagram

In this notation, the diagrammatics of adjoint functors can be drawn as follows:

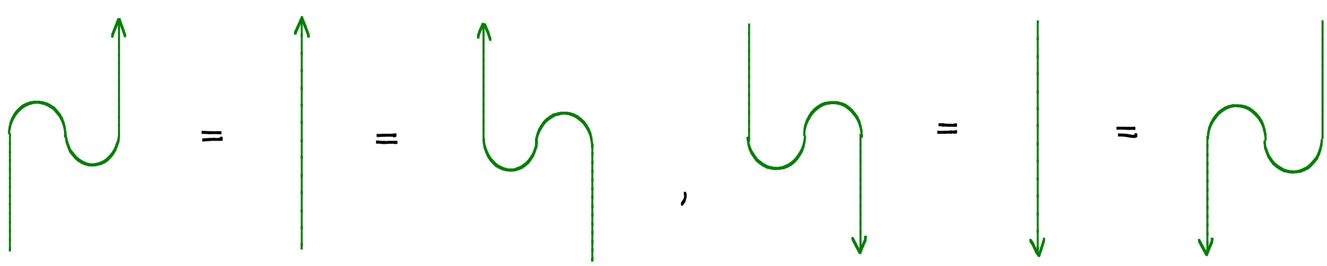




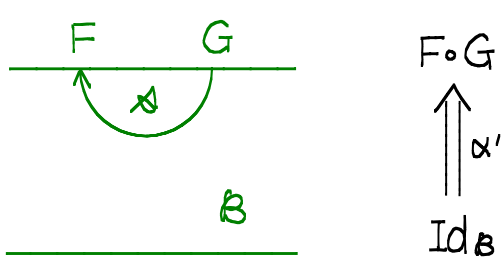
The coherence relations have the graphical interpretation:

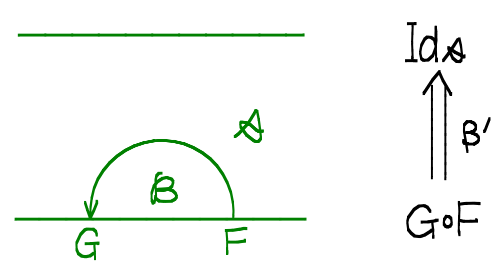


These are half of the oriented planar isotopy relations:



To get the other half, we need to require (G, F) to be an adjoint pair as well:





Def. (F, G) is a biadjoint pair if there are 2-functors α, α', β, β' satisfying the coherence relation depicted by the planar isotopy relations above.

Thus for a biadjoint pair, any planar isotopy relation of string diagrams is allowed. (String diagrams win!)

String diagrams work for any strict 2-category (in fact, weak 2-category as well). In particular, we can use it for monoidal categories (regarded as a 2-category with a unique object, whose morphisms are objects of the monoidal category, and whose 2-morphisms are morphisms of the monoidal category). As we will see in the future, it will be very useful when the 2-morphisms are given in terms of generators and relations.

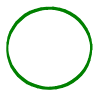
In what follows, we will give some first examples of occasions where biadjoint functors occur naturally. We will restrict our 2-category to be the 2-category of all 1-categories.

Extended $(n+2)$-d TQFT

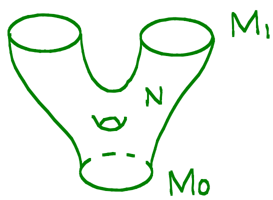
We have already seen the analogue of the concept of an extended TQFT when categorifying tangle invariants.

Def. (Extended $(n+2)$-d TQFT). An extended $(n+2)$-d TQFT is a 2-functor F from Cob_{n+2} to some 2-category (usually algebraic in nature), i.e. F assigns

Closed n-manifolds (up to diffeomorphism) $\longmapsto$ Some category


M $\longmapsto$ $F(M)$

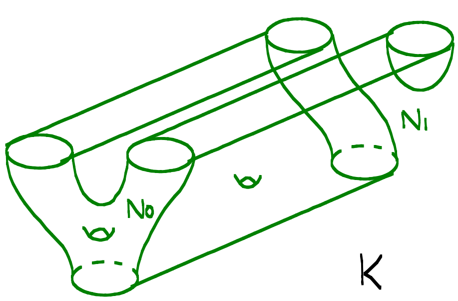
$(n+1)$-dim'l cobordisms with boundaries (up to diffeomorphism relative to the boundary) $\longmapsto$ Functors between categories



$\longmapsto$

$$\begin{tikzcd} F(M_1) \\ F(M_0) \arrow[u, "F(N)"'] \end{tikzcd}$$

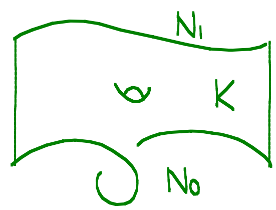
$(n+2)$-dim'l cobordisms with corners (up to diffeomorphism relative to the boundary) $\longmapsto$ Natural transformations between functors



$\longmapsto$

$$\begin{tikzcd} F(N_1) \\ F(N_0) \arrow[u, Rightarrow, "F(K)"'] \end{tikzcd}$$

For ease of drawing, in what follows, we will only draw the case $n=0$.



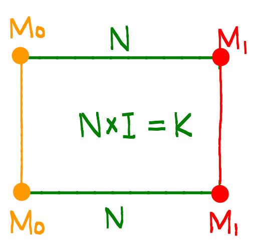
$$F(N_0) \overset{F(K)}{\Longrightarrow} F(N_1)$$

We claim that, in an extended $(n+2)$-d TQFT, any functor $F(N)$ has a natural biadjoint.

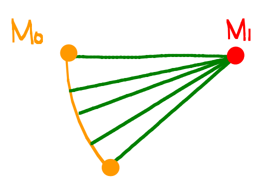
Prop 1. Let N be an $n+1$ dim'l cobordism from M_0 to M_1. Let N^* denote the cobordism N viewed backwards as a cobordism from M_1 to M_0. Then $F(N^*)$ is biadjoint to $F(N)$.

$$F(M_0) \underset{F(N^*)}{\overset{F(N)}{\rightleftarrows}} F(M_1)$$

Pf: This admits the following differential topological proof in terms of pictures:

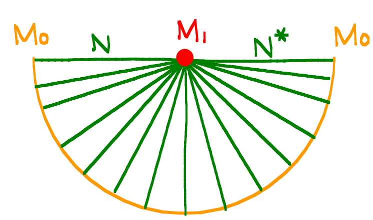


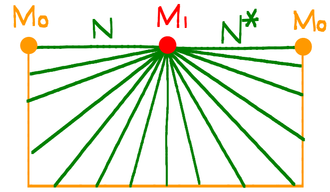
$\cong$



Relative to the boundary. Each radius is an N.

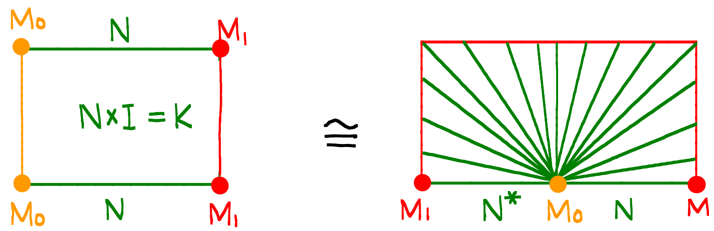
$\cong$



$\cong$ 

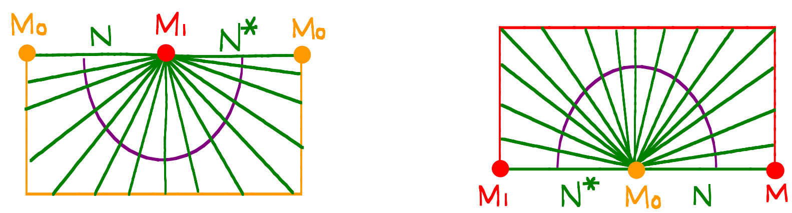
,

where the boundaries are $N^* \circ N$ and $M_0 \times I$ respectively. Hence this is a cobordism from $M_0 \times I$ to $N^* \circ N$. Similarly, one gets by first squeezing $M_0 \times I$ into a point

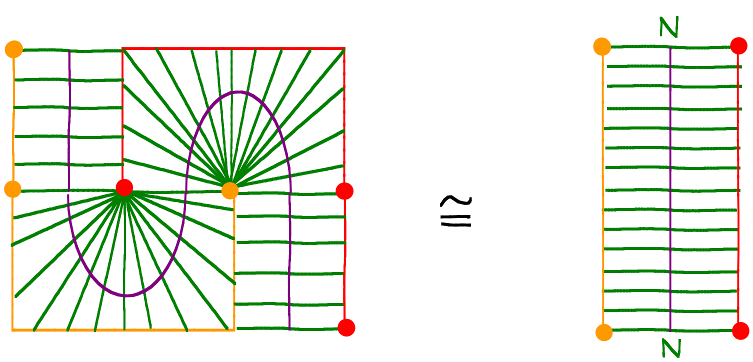


.

We regard these cobordisms as providing cups and caps:

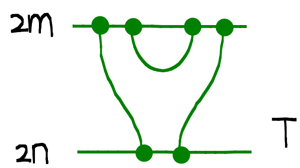


The coherence relations follow geometrically from:

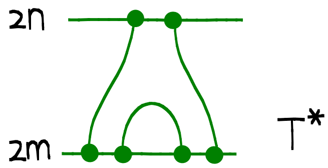


The other coherence relations follow similarly from pictures as above, and functoriality of F. The prop. follows. □

Let's reexamine the (generalised) extended TQFT from tangle cobordisms to bimodules. We assigned to tangles certain (complexes of) graded bimodules:



$$\longmapsto \quad {}_{H^m}F(T)_{H^n}$$

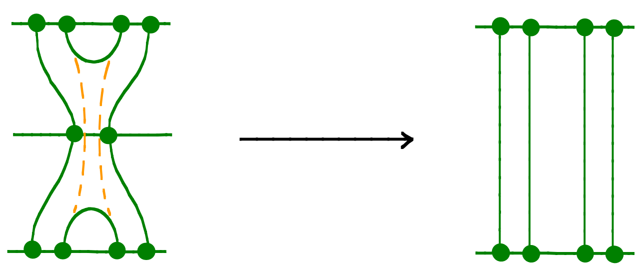


$$\longmapsto \quad {}_{H^n}F(T^*)_{H^m}$$

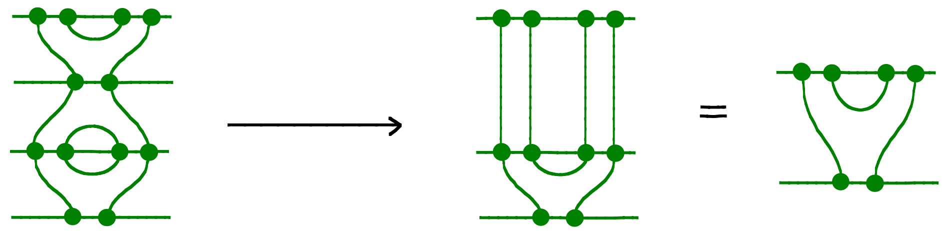
One can check that

$$\begin{tikzcd} \mathrm{Comp}(H^n\text{-mod}) \arrow[r, bend left, "F(T)\otimes_{H^n} -"] & \mathrm{Comp}(H^m\text{-mod}) \arrow[l, bend left, "F(T^*)\otimes_{H^m} -"] \end{tikzcd}$$

are bi-adjoint functors, up to a grading shift. (The grading shift reflects the fact that the functor is not quite an extended TQFT in the sense of the def. above). This follows since we used the canonical cobordism from $T^* \circ T$ to Id_m in the def. of tensor products of bimodules:



so that the composition

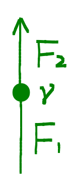


is the identity map.

One could search for extended TQFT's in categories that admit abundant biadjoint functors. Examples of such include Fukaya-Floer theory of symplectic manifolds (P. Seidel : functors between Fukaya categories Fuk(M), Fuk(N) arise as convolutions with Lagrangian submanifolds in Fuk(M×N)), or derived categories of coherent sheaves on Calabi-Yau manifolds, or representation theory. Later, we shall look back at Frobenius algebras from the view point of biadjoint functors.

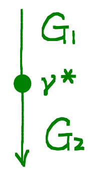
Pairs of biadjoint functors

Whenever one has more than one pair of biadjoint functors (F_1, G_1), (F_2, G_2), and a 2 functor $\gamma : F_1 \Longrightarrow F_2$, depicted by

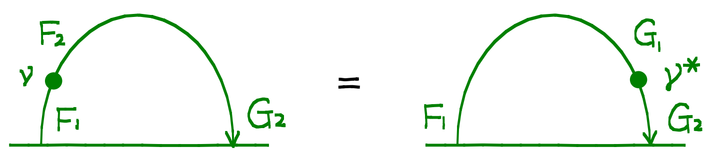


,

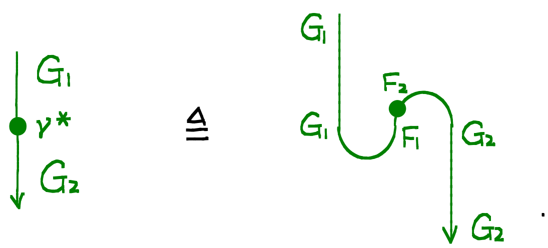
one gets a unique dual $\gamma^* : G_2 \Longrightarrow G_1$, depicted by



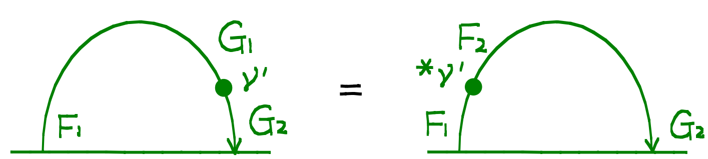
s.t.



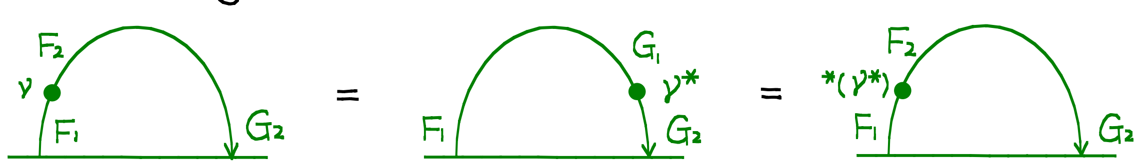
i.e. topologically, we can drag the dot along the arc. γ^* can in fact be defined just using cups and caps for the two pairs:



Likewise, any $\gamma' : G_2 \Longrightarrow G_1$ gives rise to a unique ${}^*\gamma' : F_1 \Longrightarrow F_2$ s.t.

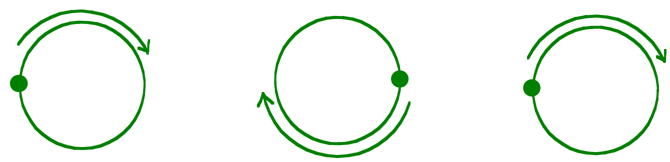


Thus, ideally, we would like to drag γ to γ^* and drag γ^* back to ${}^*(\gamma^*)$ and get the same γ back:

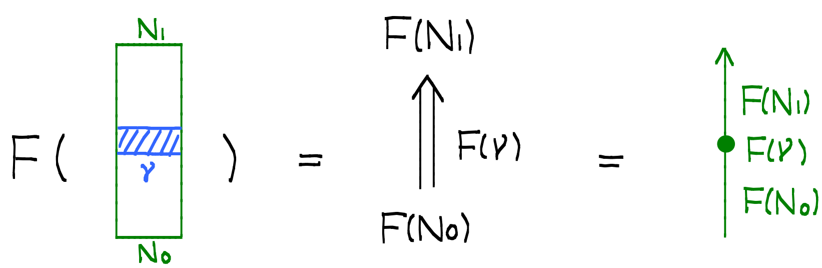


$$\nu = {}^*(\nu^*)$$

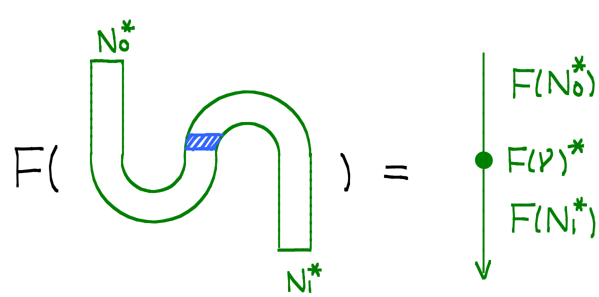
In case this happens, we shall say that the pairs of biadjunction is ambidextrous. Dots can be dragged along freely on isotopic string diagrams.



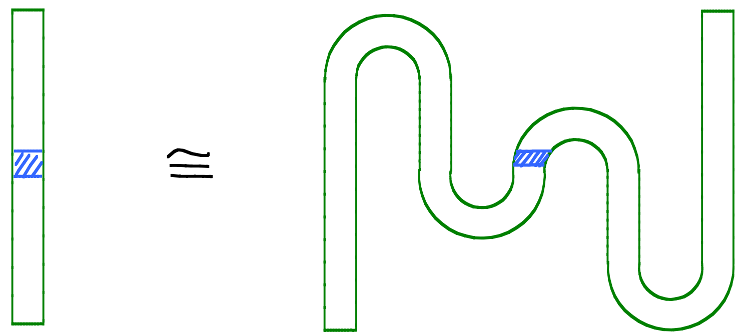
In any extended TQFT ambidextrous biadjoint pairs exist by default (via a banded version of the definition)



Then $F(\nu)^*$ is just



so that the ambidexterity condition is the functorial image of the diffeomorphism:



Ex. Think about the generalised extended TQFT of tangle cobordisms into bimodules. Does every bimodule map come from the image of a surface with boundaries and corners?

Thin surface TQFT revisited.

In general, whenever we have an inclusion of $\Bbbk$-algebras $B \subseteq A$, we have functors:

$$\begin{tikzcd} B\text{-mod} \arrow[r, bend left, "\mathrm{Ind}"] \arrow[r, bend right, "\mathrm{CoInd}"'] & A\text{-mod} \arrow[l, "\mathrm{Res}"] \end{tikzcd}$$

defined by

$$\mathrm{Ind} = {}_A A \otimes_B -$$
$$\mathrm{Res} = {}_B A \otimes_A -$$
$$\mathrm{Coind} = \mathrm{Hom}_B({}_B A, -).$$

such that

$$(\mathrm{Ind}, \mathrm{Res}), \quad (\mathrm{Res}, \mathrm{Coind})$$

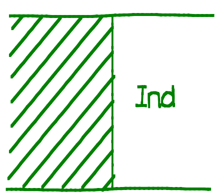
are adjoint pairs. Thus to get biadjoint pairs, it's natural to require that $\mathrm{Ind} \cong \mathrm{Coind}$.

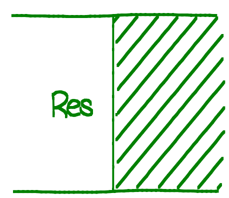
Def. The inclusion $B \subseteq A$ is called a Frobenius extension if the functors Ind and Coind are isomorphic.

(See. Kadison. New Examples of Frobenius Extensions).

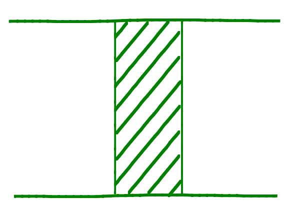
In case $B = \Bbbk$ and A is a finite dim'l $\Bbbk$-algebra, $B \subseteq A$ being a Frobenius extension is equivalent to requiring that ${}_A A \cong {}_A A^*$ (as left A-modules), which recovers our earlier notion of a Frobenius algebra (§1). We will deal with this case using string diagrams now (it works equally for any Frobenius pair).

We will depict Ind (= Coind), Res functors respectively by:





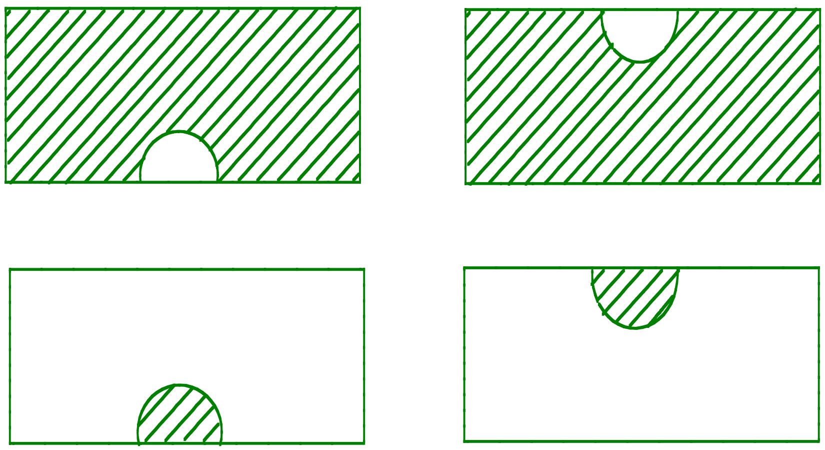
where the shaded region denotes the category of A-modules, while the unshaded denotes that of $\Bbbk$-modules. Thus the picture



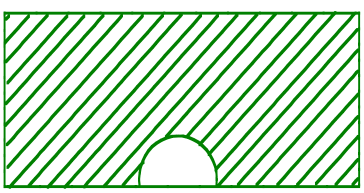
denotes the functor

$$\begin{aligned} \mathrm{Res} \circ \mathrm{Ind} &\cong {}_{\Bbbk}A \otimes_A A \otimes_{\Bbbk} - \\ &\cong {}_{\Bbbk}A_{\Bbbk} \otimes - \end{aligned}$$

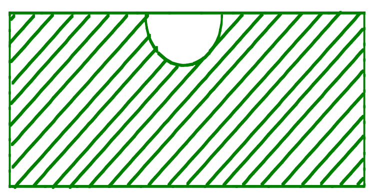
Since this functor recovers A as the $\Bbbk$-vector space $\mathrm{Res} \circ \mathrm{Ind}(\Bbbk)$, we will just identify A with this band. Then the adjunctions are depicted:



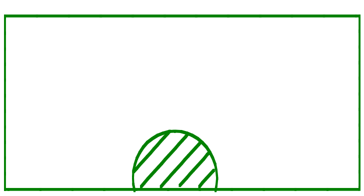
which can be identified with the functors:



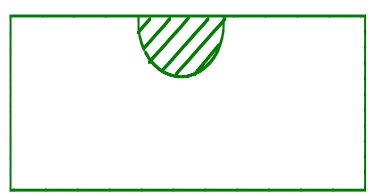
$$\begin{array}{c} A(\otimes_A -) \\ \uparrow m \\ A \otimes_{\Bbbk} A(\otimes_A -) \end{array}$$



$$\begin{array}{c} A \otimes_{\Bbbk} A \\ \uparrow \Delta \\ A \end{array}$$

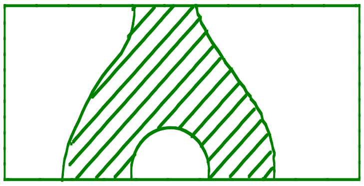


$$\begin{array}{c} \Bbbk \\ \uparrow \epsilon \\ A \end{array}$$

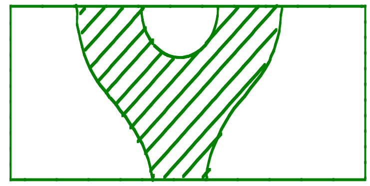


$$\begin{array}{c} A \\ \uparrow \iota \\ \Bbbk \end{array}$$

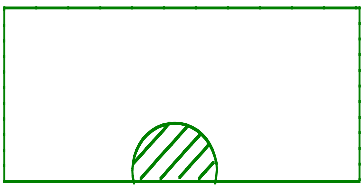
By some further induction from / restriction to $\Bbbk$-modules, one recovers our thin surfaces and the corresponding maps of $\Bbbk$-vector spaces:



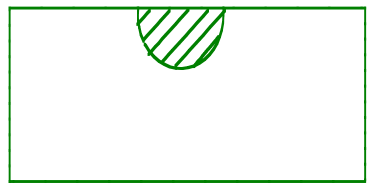
$$ {}_{\Bbbk}A \otimes_{\Bbbk} A_{\Bbbk} \xrightarrow{m} {}_{\Bbbk}A_{\Bbbk} $$



$$ A_{\Bbbk} \xrightarrow{\Delta} {}_{\Bbbk}A \otimes_{\Bbbk} A_{\Bbbk} $$

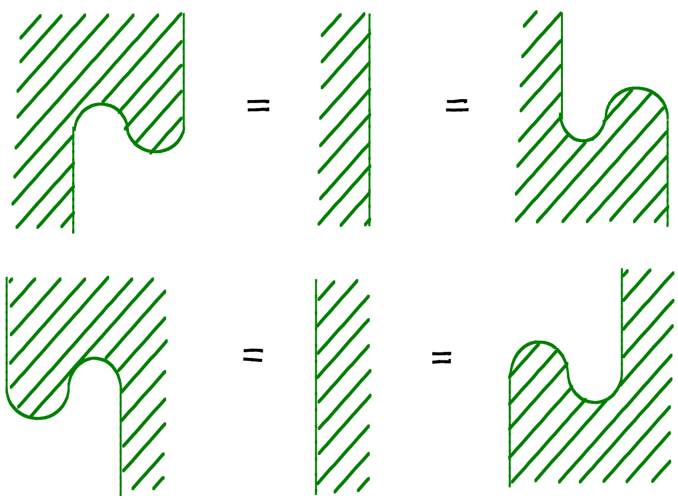


$$ {}_{\Bbbk}A_{\Bbbk} \xrightarrow{\epsilon} \Bbbk $$



$$ \Bbbk \xrightarrow{\iota} {}_{\Bbbk}A_{\Bbbk} $$

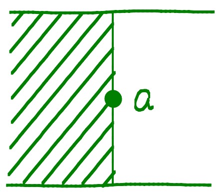
The biadjointness of Ind, Res implies that

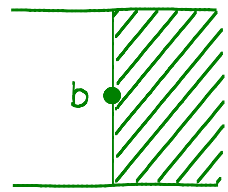


which in turn shows that the above operations $(m, \Delta, \epsilon, \iota)$ make the $\Bbbk$-vector space A into a Frobenius algebra.

We conclude from this discussion that the thin surface TQFT description of a Frobenius algebra is none other than the string diagram depiction of the Frobenius pair (Ind, Res).

Lastly, let's look at endomorphisms of the biadjoint pair (Ind, Res). We will depict them by a dot carrying certain label an endomorphism of Ind or Res:





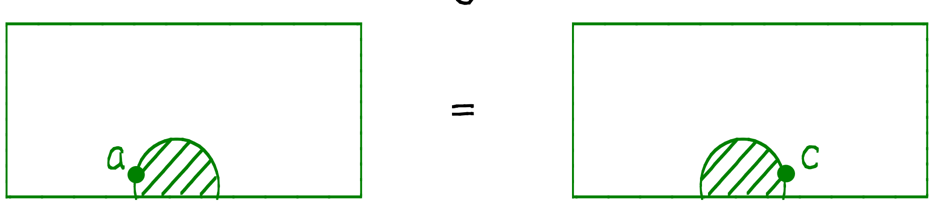
Such an endomorphism, say, of $\mathrm{Ind} = {}_A A \otimes_{\Bbbk} -$, is left A-linear. Hence it's totally determined by the image of $1 \in {}_A A$, i.e. it's induced by

$$\begin{array}{c} \mathrm{Ind} = {}_A A \otimes_{\Bbbk} - \\ \Big\Uparrow a \\ \mathrm{Ind} = {}_A A \otimes_{\Bbbk} - \end{array}$$

the right multiplication of A by the image of 1. We will also call this element a. Conversely, any $a \in A$ gives rise to an endomorphism of Ind in this fashion.

Likewise, any endomorphism of Res comes from left multiplication by an element of A.

Then one can ask about duals of endomorphisms:



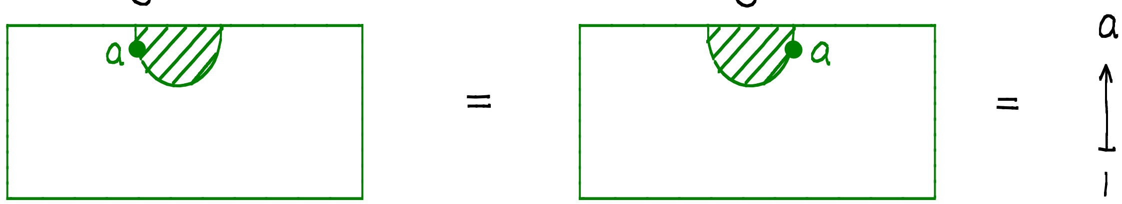
$c = ?$

This is just the same as asking for an element $c \in A$ s.t. $\forall b \in A$,

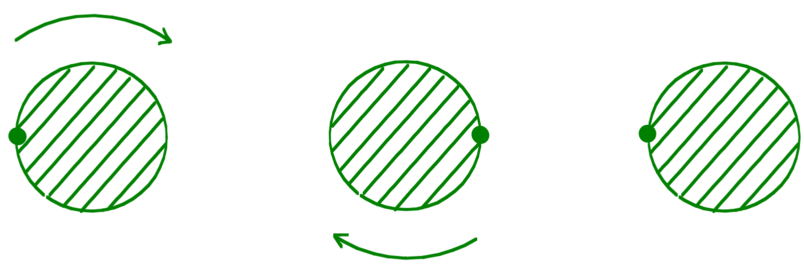
$$\epsilon(ab) = \epsilon(bc)$$

But recall that this is just the def. of the Nakayama automorphism τ of A: $\forall b \in A$, $\epsilon(ab) = \epsilon(b\tau(a))$, so that $c = \tau(a)$.

The dual of a under ι remains unchanged:



Then to have ambidextrous biadjuctions, we need to be able to drag dots freely on lines:



$$a \longmapsto \tau(a) \longmapsto a,$$

which means that $\tau = \mathrm{id}_A$, i.e. A is symmetric.

All the above discussion recovers our earlier result that the image of a thin surface TQFT is a symmetric Frobenius algebra.