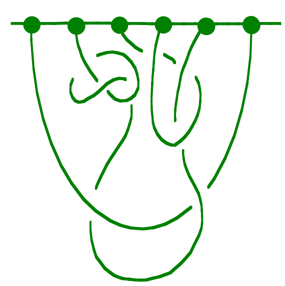
§5. Extending Link Homology To Tangles

Previously in §3 we defined the graded Frobenius algebra $A \cong \mathbb{Z} \oplus \mathbb{Z}x$ to lift the Jones polynomial to a graded chain complex up to homotopy and we showed that it's a link invariant. In this section we shall extend this "lifting process", or categorification of the Jones polynomial to the case of tangles.

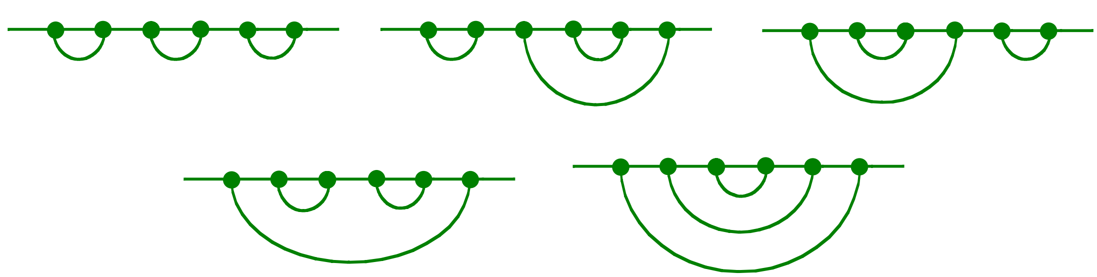
The ring H^n

Recall that we used the set of matchings when localizing links into tangles. Now we use the same process for tangles with boundary points fixed on a line segment:

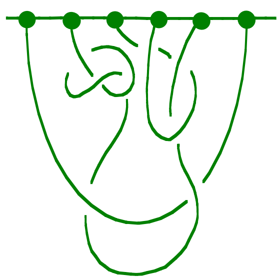


We introduce the set B_n of matchings of $2n$ points just as before ($2n$ points on a circle):

E.g. The 5 matchings in B_3:



Rmk: Our aim is to define a functorial tangle invariant so tangles as above should be regarded as a morphism (1-morphism) from the object ϕ to $2n$ points.



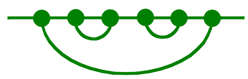
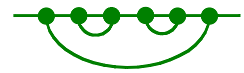
$T \xmapsto{\ F\ }$

$$\begin{tikzcd} \text{Certain } \mathbb{Z}[q,q^{-1}]\text{-module } K_0(C_n) \\ \mathbb{Z}[q,q^{-1}] = K_0(C_\phi) \arrow[u, "F(T)"'] \end{tikzcd}$$

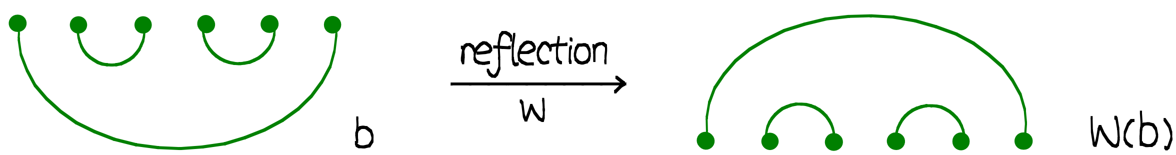
Thus we need to specify categories C_n for each $2n$ points. Tangles will be sent to functors between C_n and C_m, while tangle cobordisms will be sent to natural transformations between functors. To match with our previous categorification of the Jones polynomial we will need to resolve tangles as before by matchings in B_n (with possible O's inside), and apply the same TQFT we used in §3. First off, we need $K_0(C_n)$ to be free $\mathbb{Z}[q,q^{-1}]$-modules parametrized by B_n. Then we will figure C_n out by taking into account Hom spaces of C_n's.

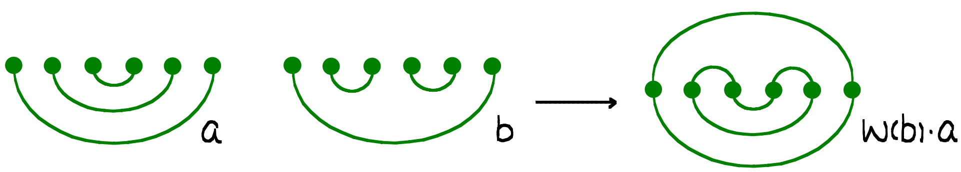
Similar to §3, we introduce the free $\mathbb{Z}[q,q^{-1}]$-module I^N with a basis given by matchings of $2n$ points on a line segment (so $N = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan numbers).

Thus to each matching $a \in B_n$, we have associated with it a free $\mathbb{Z}[q,q^{-1}]$-module Q_a

$a =$  $\longmapsto$ $Q_a = \mathbb{Z}[q,q^{-1}]\langle$  $\rangle$

Then what should we assign to $\mathrm{Hom}(a,b)$? We shall reflect b about the horizontal axis, glue it on top of a, and apply out TQFT F:





$\xrightarrow{F}$ $F(W(b)a) \cong A^{\otimes 2}$

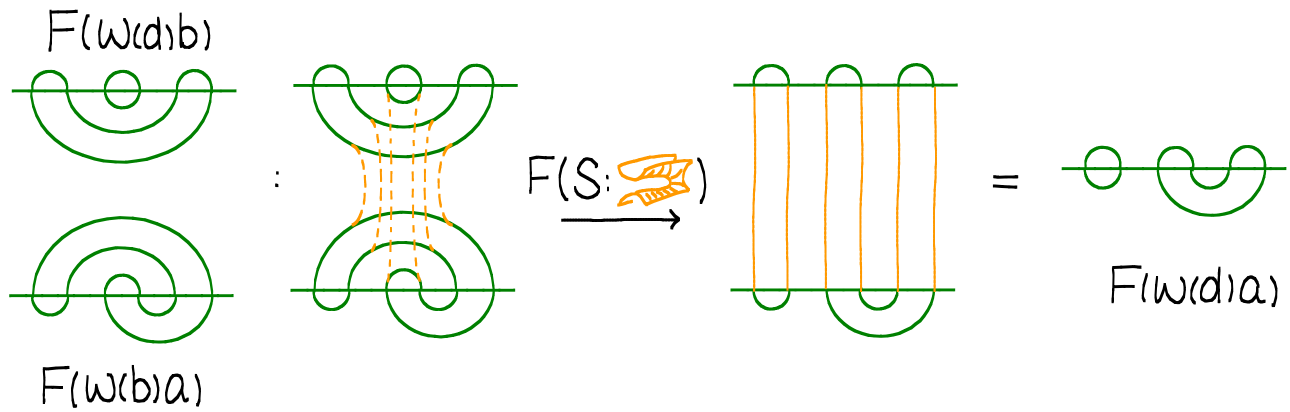
Def. (The ring H^n) For each $n \in \mathbb{Z}_{\geq 0}$, we define a finitely generated graded abelian group H^n to be

$$H^n \triangleq \bigoplus_{a,b\in B_n} F(W(b)\cdot a)\{n\}$$

We will explain the grading shift later. And first of all, we define a multiplication on H^n, bearing in mind the general principles of TQFT's from §2.

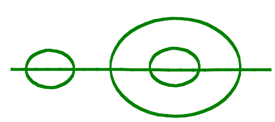
$$F(W(d)c) \otimes F(W(b)a) \longrightarrow \delta_{b,c}\cdot F(W(d)a)$$

The map will be 0 unless $b=c$. (Imagine the product given by matrix multiplication with rows/columns numbered by matchings). When $b=c$, the product is given by evaluating F on the canonical cobordism of the saddle moves that merge circles:



The associativity of this multiplication follows immediately: different orders of multiplications are given by composition of different orders of saddle moves that are far away so that they are isotopic.

Next, observe that $W(a)\cdot a$ consists exactly of n circles for any $a \in B_n$:



$W(a)\cdot a$

so that $1^{\otimes n} \in F(W(a)\cdot a) \cong A^{\otimes n}$. We shall denote this element by 1_a. Furthermore, multiplication by 1_a is a projection

$$\begin{array}{ccccc} F(W(d)c) & \otimes & F(W(a)a) & \longrightarrow & \delta_{c,a} F(W(d)a) \\ y & & 1_a & \longmapsto & \delta_{c,a}\, y \end{array}$$

This is because, in our 2d-TQFT F, merging a circle into the others is given by multiplication of A's, and 1_a consists of units for this multiplication.

Similarly, one shows that 1_a's are projections on the left as well. It follows that 1_a's are idempotents in H^n. From this we see that the sum of 1_a's, when a ranges over B_n, gives the identity element of H^n:

$$1_{H^n} = \sum_{a \in B_n} 1_a$$

This discussion also shows the necessity of the grading shift in the definition of H^n: 1_{H^n} now sits in q-degree 0 (recall that in A, 1 sits in q-degree -1, so that $1_a = 1^{\otimes n} \in A^{\otimes n}$ sits in q-degree $-n$).

The existence of these idempotents also lets us formally decompose H^n into a matrix ring:

$$H^n \cong \bigoplus_{a,b \in B_n} 1_b H^n 1_a$$

$$= \bigoplus_{a,b \in B_n} F(W(b)a)$$

which makes our earlier comparison of H^n with a matrix algebra legitimate.

$$H^n \cong \left\{ \begin{array}{cc} & a \\ W(b) & \left(\begin{array}{ccc} & \vdots & \\ \cdots & F(W(b)a) & \cdots \\ & \vdots & \end{array} \right) \end{array} \right\}$$

In summary, we have shown that

Thm 1. H^n is a graded, associative, unital ring. $\square$

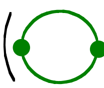
E.g. H^n for $n = 0, 1, 2$.

(1). $n=0$. We only have empty diagrams, so:

$$H_0 \cong F(W(\emptyset)\emptyset) \cong \mathbb{Z}$$

(2). $n=1$. There is only 1 matching:

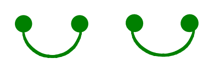
so that

$H^1 = F($  $)\{1\} = A\{1\} \cong \mathbb{Z}[X]/(X^2)$

with its usual ring structure.

(3). $n=2$. Now there are two matchings in B_2:

a

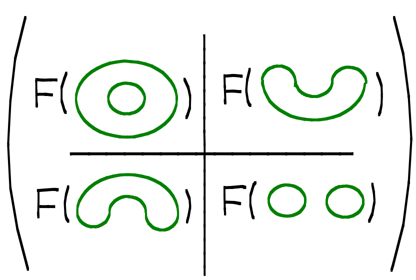


b

Thus H^2, written as a matrix algebra, has the form:

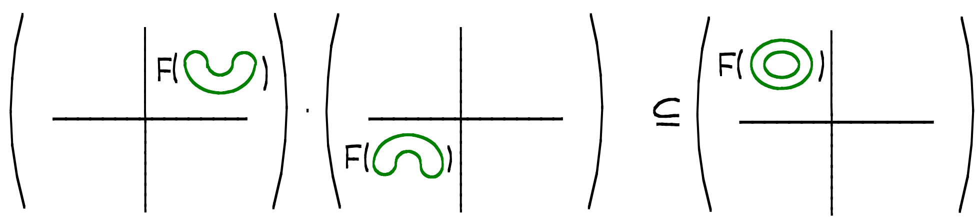
$$H^2 \cong \left(\begin{array}{c|c} F(W(a)a) & F(W(b)a) \\ \hline F(W(a)b) & F(W(b)b) \end{array}\right)$$

$\cong$

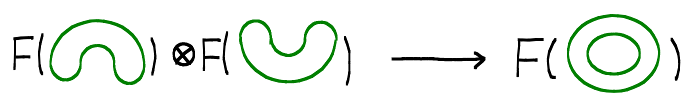


$$\cong \left(\begin{array}{c|c} A^{\otimes 2} & A \\ \hline A & A^{\otimes 2} \end{array}\right)$$

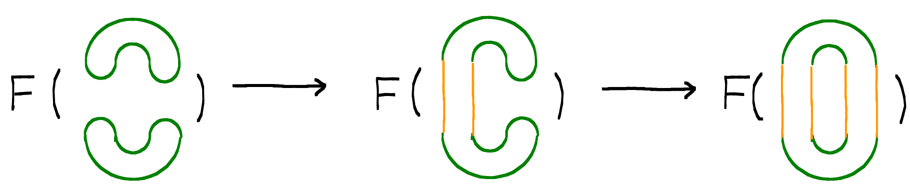
The multiplication is defined using multiplication and comultiplication of A, for instance,



where the entry product



can be determined by first decomasing the map into



$1 \otimes 1$	$\longmapsto$	1	$\longmapsto$	$1 \otimes x + x \otimes 1$
$1 \otimes x$	$\longmapsto$	x	$\longmapsto$	$x \otimes x$
$x \otimes 1$	$\longmapsto$	x	$\longmapsto$	$x \otimes x$
$x \otimes x$	$\longmapsto$	0	$\longmapsto$	0

The other products in H^2 can be similarly computed

Rmks: (1). H^n also admits a description as a path algebra modulo relations. However, it's not clear this description is useful.
(2). H^n-f.g. mod is Morita equivalent to a certain parabolic category $\mathcal{O}$ of $\mathfrak{U}_n$
(3). H^n is also Morita equivalent to some $\Bbbk[G]$ for some finite group G where $\mathrm{char}(\Bbbk) \mid |G|$.

$K_0(H^n)$ and $G_0(H^n)$

From now on we shall pass from $\mathbb{Z}$ to a field $\Bbbk$. We will be working with $H^n \otimes_{\mathbb{Z}} \Bbbk$, which we still denote by H^n.

The idempotents 1_a, $a \in B_n$ decompose H^n into projectives:

$$H^n \cong \bigoplus_{a \in B_n} H^n \cdot 1_a \triangleq \bigoplus_{a \in B_n} P_a$$

and we claim that

Prop 2. $\{P_a = H^n \cdot 1_a \mid a \in B_n\}$ is a complete list of indecomposable graded projective modules, which are pairwise non-isomorphic.

To prove this, we recall some general facts about graded rings and their Grothendieck groups of suitable (f.d. / f.g. / f.l. etc.) graded modules.

If R is such a ring, we have an automorphism of its graded module category $\{1\}$:

$$\begin{aligned} \{1\}: \mathrm{gr.}\ R\text{-mod} &\longrightarrow \mathrm{gr.}\ R\text{-mod} \\ M &\longmapsto M\{1\} \end{aligned}$$

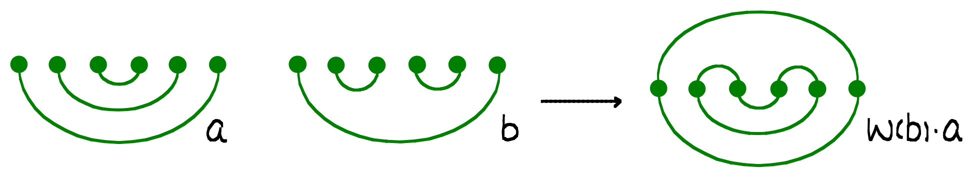
In this way the Grothendieck groups of suitable graded module categories $\mathcal{A}$ become a $\mathbb{Z}[q, q^{-1}]$-module:

$$\begin{aligned} q^{\pm 1}: G_0(\mathcal{A}) &\longrightarrow G_0(\mathcal{A}) \\ [M] &\longmapsto [M\{\pm 1\}] \end{aligned}$$

Proof of Prop. 2.

We need to find out the Jacobson radical of H^n. Note that with the grading chosen, H^n is $\mathbb{Z}_{\geq 0}$-graded and the multiplication is grading preserving.

All the "off-diagonal" terms are strictly positively graded. This is because when $a, b \in B_n$, $a \neq b$, $W(a)b$ has strictly fewer than n circles:



Then the grading shift $\{n\}$ in the definition of H^n shifts all the elements in $F(W(b) \cdot a)$ into strictly positive q-degrees.

On the other hand, on the diagonal, the lowest degree term is $1_a = 1^{\otimes n} \in F(W(a) \cdot a) \cong A^{\otimes n}$ (recall that x has degree 1 in A), and thus $\{n\}$-shift makes it the only degree 0 term in $F(W(a) \cdot a)$.

It follows that $J(H^n) = (H^n)_{>0}$ and

$$H^n / J(H^n) \cong \bigoplus_{a \in B_n} \Bbbk \cdot 1_a$$

This in turn implies that $\{1_a \mid a \in B_n\}$ is a complete list of indecomposable idempotents. The result follows.

$\square$

Notation: For any graded module M over a graded ring and $f = \sum_{a\in\mathbb{Z}} m_a q^a \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$, we shall write

$$M^{\oplus f} \triangleq \bigoplus_{a\in\mathbb{Z}} M\{a\}^{\oplus m_a}$$

Cor. 3. For graded projective H^n-modules, we have

$$K_0(H^n) \cong \bigoplus_{a\in B_n} \mathbb{Z}[q, q^{-1}]\cdot[P_a] \cong I^N$$

$\square$

Cor. 4. The graded simple finite length modules are exactly the

$$L_a \cong \Bbbk\cdot 1_a$$

for any $a \in B_n$. Thus $\mathrm{End}_{H^n}(L_a) \cong \Bbbk$.

$$G_0(H^n) \cong \bigoplus_{a\in B_n} \mathbb{Z}[q, q^{-1}]\cdot[L_a] \cong I^N$$

and the pairing between K_0 and G_0 (see §4.)

$$\begin{aligned} K_0(H^n) \otimes_{\mathbb{Z}[q,q^{-1}]} G_0(H^n) &\longrightarrow \mathbb{Z}[q, q^{-1}] \\ (P \quad , \quad M) &\longmapsto \dim_{\Bbbk} \mathrm{HOM}(P, M) \\ &\triangleq \dim_{\Bbbk}\left(\bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}(P\{k\}, M)\right) \end{aligned}$$

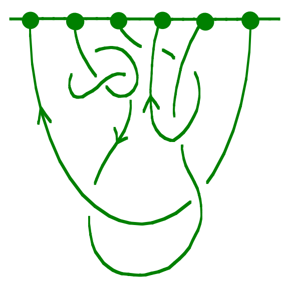
is a perfect pairing of $\mathbb{Z}[q,q^{-1}]$-modules. $\square$

Rmk: Observe that simply by restricting to graded modules we essentially trivialized the problem of calculating K_0 and G_0. On the other hand, if we remove the grading constraint, the problem will be much harder!

Meta-Problem: Think about using higher K-theory in problems arising from categorifications.

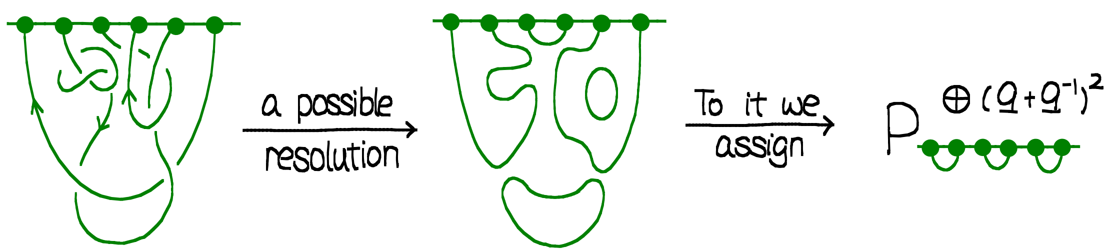
Invariant of tangles: the usual approach

Now we briefly sketch how we can obtain oriented $(0,2n)$-tangle invariants of chain complexes as we did for oriented links



A $(0,6)$-tangle

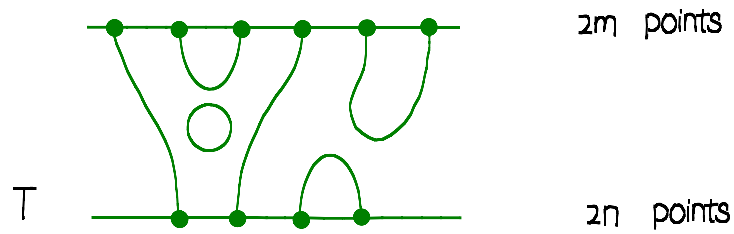
To any such a tangle, we take its complete resolution into $2^{\#(\text{crossing})}$-matching diagrams (with embedded circles), together with resolution cobordisms between them.



Collapsing this multicube of maps of projective modules and add appropriate - signs, we get a chain complex of projective H^n modules $C'(D)$. After some grading and homological degree shifts we obtain another chain complex of projective H^n module $C(D)$. One procede as before to show the invariance of $C(D)$ under tangle moves.

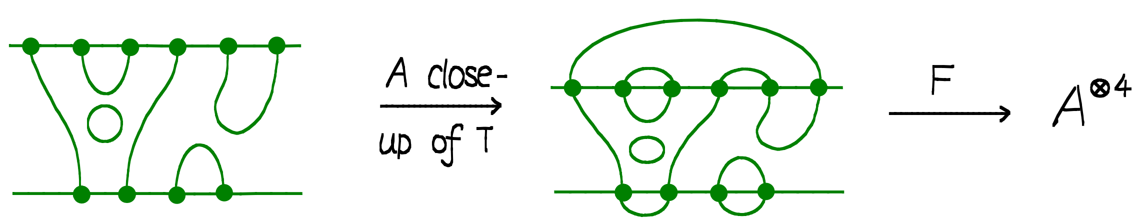
Invariant of tangles: Flat tangles and H^n-bimodules

In this subsection we shall construct tangle invariants using H^n-bimodules. We first focus on flat (m,n)-tangles on the plane:

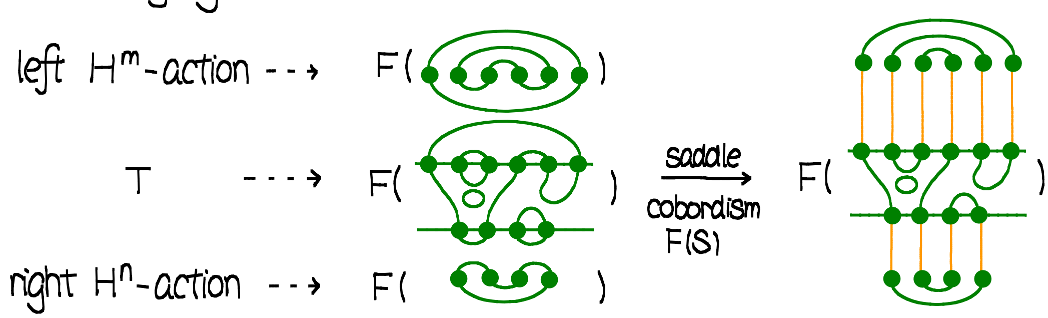


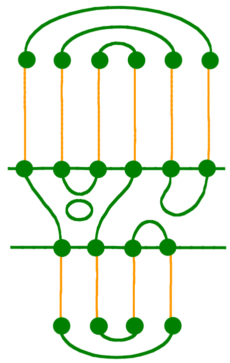
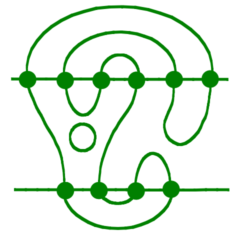
To such a tangle T we shall associate with it an (H^m, H^n) projective bimodule $F(T)$. It's defined by taking all possible close-up's of T using reflected images of $2m$-matchings on top of T, and using $2n$-matchings of T on the bottom. Then we apply our TQFT F to it and shift the grading by n:

$$F(T) \triangleq \bigoplus_{a\in B_n,\, b\in B_m} F(W(b)Ta)\{n\}$$



In this way, $F(T)$ becomes graded left H^m, right H^n bimodule where the module structure is defined using saddle cobordisms and applying our TQFT F:



$F($  $)$ $=$ $F($  $)$ $= A^{\otimes 2}$

i.e. the module map

$$(H^m, F(T), H^n) \longrightarrow F(T)$$

is determined by the cobordism's S.

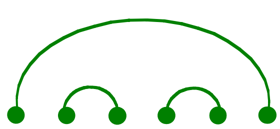
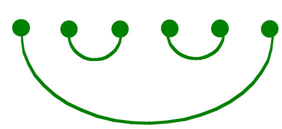
E.g. Some easy $F(T)$'s

(1). In the special case where $T = | \; | \; | \cdots |$, we get

$$F(T) \cong {}_{H^n}H^n{}_{H^n}$$

as an (H^n, H^n)-bimodule.

(2). In case $T = a \in B_n$ or $W(a) \in W(B_n)$:

$T=$  and $T'=$ 

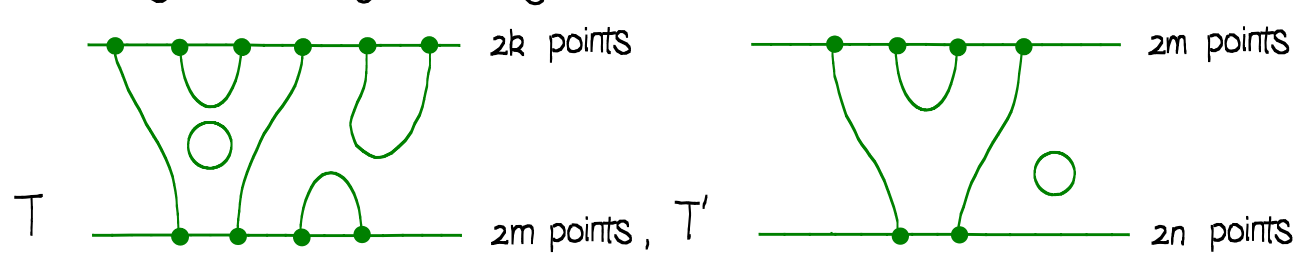
we get $F(T) \cong 1_a \cdot H^n \cong {}_aP$, and $F(T') \cong H^n \cdot 1_a \cong P_a\{-n\}$

In general, as a bimodule ${}_{H^m}F(T)_{H^n}$ is both projective as a left H^m-module and as a right H^n-module.

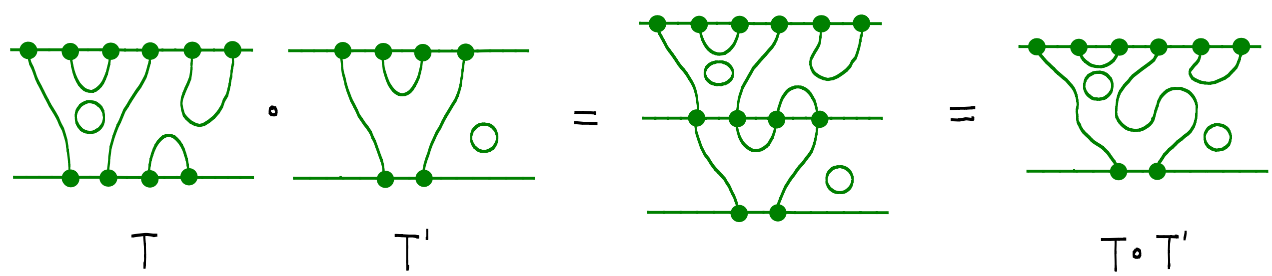
There are two main points of this bimodule assignment being a functorial flat tangle invariant:

(1). Composition of flat tangles.

Now given two flat tangles T, T':



we can compose them whenever it makes sense:



We have defined for T, T', $T\circ T'$ bimodules ${}_{H^k}F(T)_{H^m}$, ${}_{H^m}F(T')_{H^n}$, ${}_{H^k}F(T\circ T')_{H^n}$, respectively. Consider the map

$$\begin{array}{ccc}
{}_{H^k}F(T)_{H^m} \otimes_{I_k} {}_{H^m}F(T')_{H^n} & \longrightarrow & {}_{H^k}F(T\circ T')_{H^n} \\
\wr\| & & \wr\| \\
\bigoplus_{\substack{d\in B_k,\, a\in B_n \\ b,c\in B_m}} F(W(d)Tc)\otimes_{I_k} F(W(b)T'a) & \longrightarrow & \bigoplus_{d\in B_k,\, a\in B_n} F(W(d)T\circ T'a)
\end{array}$$

It descends to a map of projective (H^k, H^n)-bimodules,

$$_{H^k}F(T)_{H^m} \otimes_{H^m} {}_{H^m}F(T')_{H^n} \longrightarrow {}_{H^k}F(T \circ T')_{H^n}$$

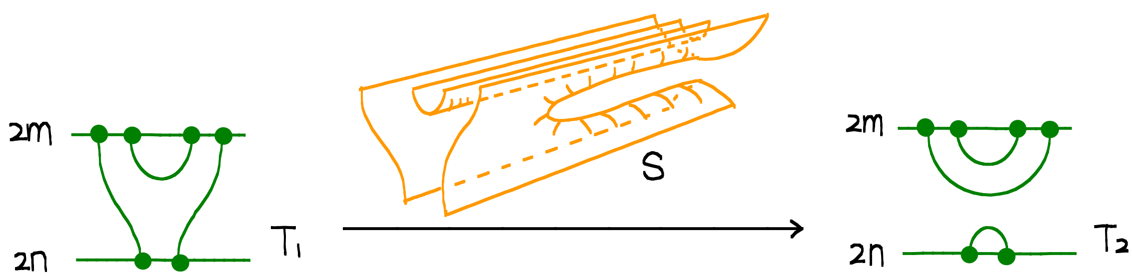
and it's not hard to show that

Thm 5.
$$F(T) \otimes_{H^m} F(T') \cong F(T \circ T')$$

as (H^k, H^n)-bimodules.

This essentially follows from the functoriality of F as a TQFT
Proof omitted. See M. Khovanov, A Functor Valued-Invariant of Tangles.

(2). Functoriality of tangle cobordisms.
A flat tangle cobordism between T, T' of the same boundary points is an embeded surface S



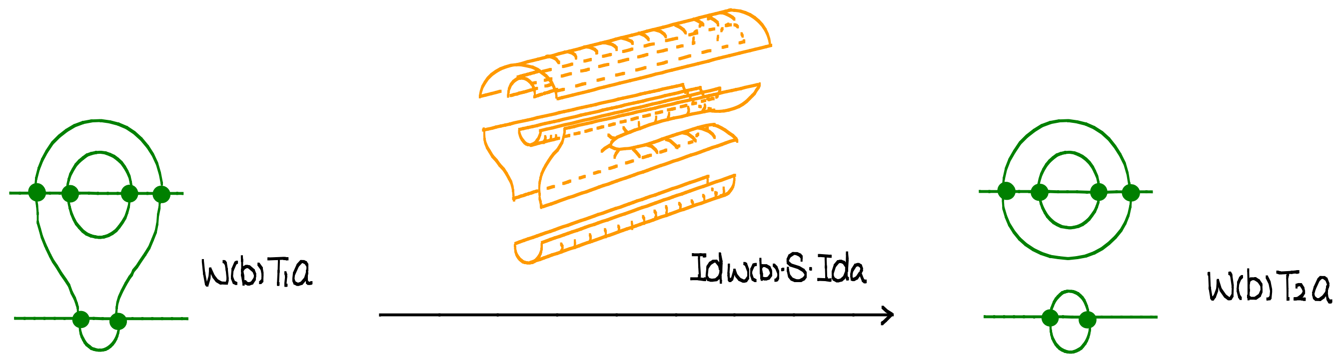
with the boundary condition that

$$\partial S = T_1 \cup T_2 \cup (\partial T_1 \times I)$$

To such a cobordism S we shall associate a map of bimodules

$$F(S) : F(T_1) \longrightarrow F(T_2).$$

We "cup" and "cap" off S using all possible matchings time I, which gives rise to surfaces $Id_{W(b)} \cdot S \cdot Id_a$ as cobordisms between $W(b)T_1a$ and $W(b)T_2a$:



To such a cobordism, we can apply our TQFT F, to get a map

$$F(Id_{W(b)} \cdot S \cdot Id_a) : F(W(b)T_1a) \longrightarrow F(W(b)T_2a)$$

Summing over all possible a, b's, we get an (H^m, H^n) bimodule map which we denote by $F(S)$:

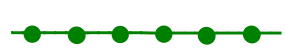
$$F(S) = \bigoplus_{a \in B_n, b \in B_m} F(Id_{W(b)} \cdot S \cdot Id_a) : F(T_1) \longrightarrow F(T_2)$$

It's readily seen that $F(S)$ is a homogeneous map whose degree is

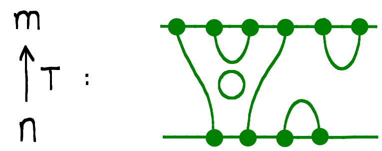
$$\deg F(S) = -\chi(S) + n + m$$

We summarize what we have "lifted" so far. We define a 2-category Flat 2-Tangle as:

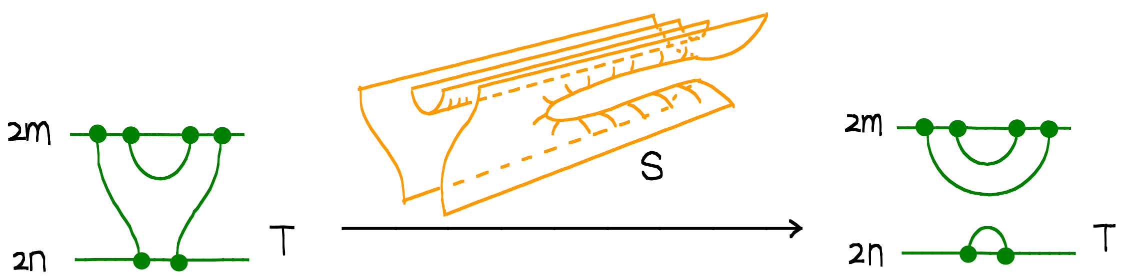
Objects: $\mathbb{Z}_{\geq 0}$. To each $n \in \mathbb{Z}_{\geq 0}$, we assign $2n$ points on a line



1-morphisms : $1\text{-Mor}(n,m) \triangleq \{$ flat tangles $\subseteq \mathbb{R}\times I$, of $2n$ bottom boundary points, and $2m$ top boundary points $\}$.



2-morphisms : For any $T, T' \in 1\text{-Mor}(n,m)$, $2\text{-Mor}(T,T') \triangleq \{$ surface cobordisms S between T, T' in $\mathbb{R}\times I\times I$, fixing boundary conditions of T and T' $\}$.



Our construction above is then a 2-functor :

$$
\begin{array}{llll}
F: & \text{Flat 2-Tangles} & \longrightarrow & \text{Bimodules} \\
 & n & \longmapsto & H^n \\
 & T & \longmapsto & F(T) : (H^m, H^n)\text{-bimodule} \\
 & S & \longmapsto & F(S) : \text{bimodule maps}
\end{array}
$$

This is in some sense a TQFT with corners, and with restriction to 3-d ambient spaces.

Notice that all the constructions above only depended on F being a 2d-TQFT but not the ring $A \cong \mathbb{Z}[X]/(X^2)$, so that we can replace A by any commutative Frobenius algebra A over a ground ring $\mathbb{k}$. Later we will see that it is the requirement that F be a link invariant that restricts $A/\mathbb{k}$ to be relatively small. And we will see some variations using other $A/\mathbb{k}$.

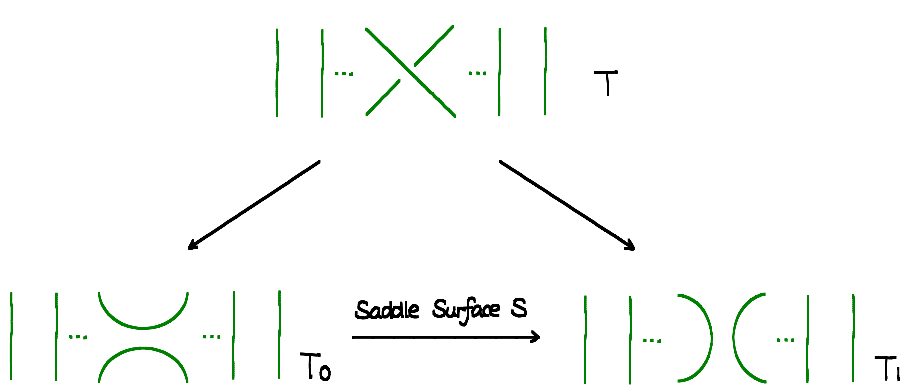
Problem: Can one extend the above story to all (oriented) matchings of $2n$ points ($n!$ many of them):

One would then get a 2-functor restricted to 4d (note that the tangles sit in $\mathbb{R}^2 \times I$, so cobordisms S between tangles lie in $\mathbb{R}^2 \times I \times I \subseteq \mathbb{R}^4$).

Invariant of tangles: all tangles

Once we have constructed flat tangle invariants as in the previous subsection, we can extend it to all tangles using the same procedure as for the Jones polynomials.

First, for the unoriented tangle of a crossing, we take its resolutions:



and then assign to it the cone of the bimodule map:

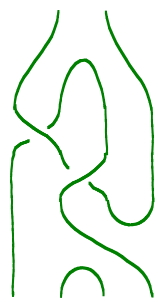
$$F(S): \; F(T_0) \longrightarrow F(T_1)$$

namely, we define $F'(T)$ to be the chain complex of (H^n, H^n)-bimodules:

$$F'(T): \cdots \longrightarrow 0 \longrightarrow F(T_0) \xrightarrow{F(S)} F(T_1)\{-1\} \longrightarrow 0 \longrightarrow \cdots$$

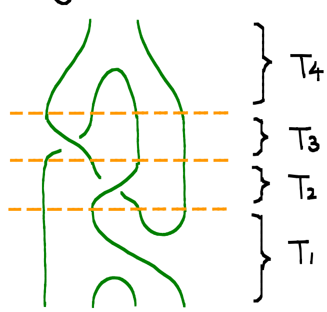
where $F(T_0)$ sits in homological degree 0, and we shift the q-degree of $F(T_1)$ down by 1, as for the Jones polynomial.

For composition of unoriented tangles:



T

we decompose it into composition of flat tangles and crossings, and assign to it the tensor product of chain complexes:



$$\longmapsto \quad F(T_4) \otimes_{H^k} F(T_3) \otimes_{H^m} F(T_2) \otimes_{H^n} F(T_1)$$

Finally, for a tangle diagram that is oriented, we count the number of over and under crossings, and shift the q-deg. and homological deg. as in the Jones polynomial case:

$$F(T) \triangleq F'(T)[x(T)]\{2x(T)-y(T)\}$$

We denote the chain complex of (H^n, H^m)-bimodules so obtained by $F(T)$ as well.

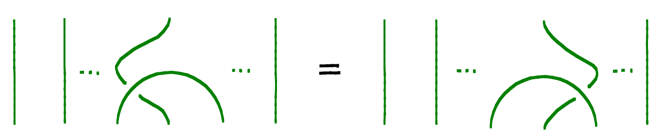
We have the following:

Thm 6. 1). $F(T)$ is an invariant of tangles in the homotopy category of chain complexes of (H^m, H^n)-bimodules.

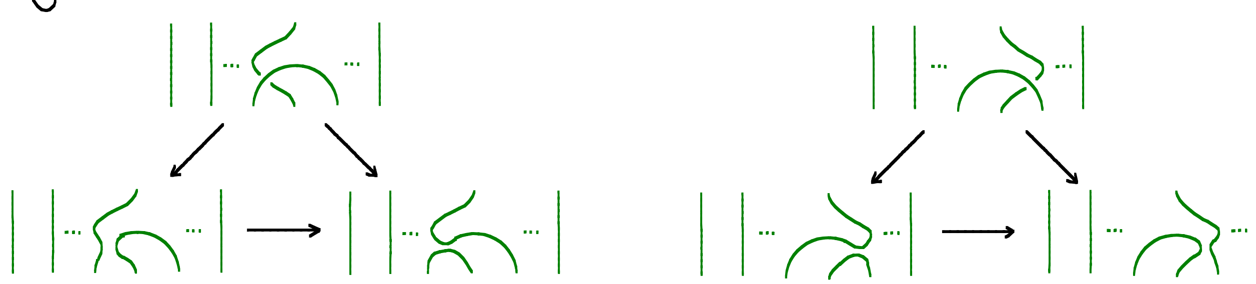
2). $F(T)$ extends to a 2-functor from the 2-category of tangle cobordisms to the 2-category of chain complexes of bimodules, up to a sign (see the remark at the end).

Sketch of proof of thm. 6.

Part 1) of the thm follows by identical arguments as in the Jones polynomial case, except that we need to check 1 more invariance under the tangle move:

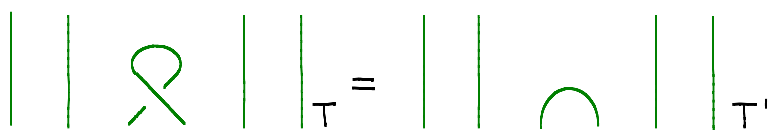


This follows easily since after resolution, both sides give isotopic diagrams:

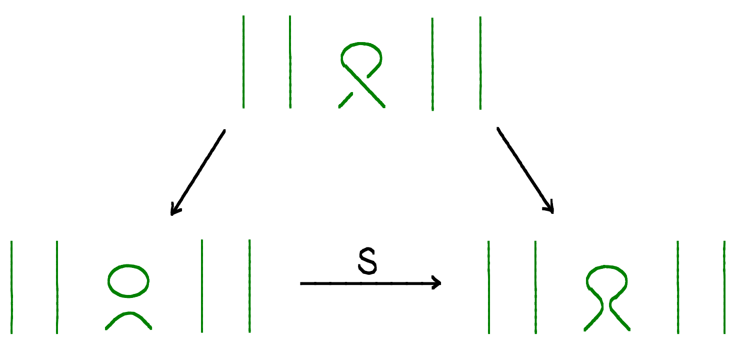


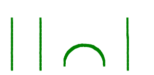
and the invariance follows from functoriality of the 2d TQFT we started with.

The remaing Redemeister moves I, II, III follow just as for the Jones polynomial. We show RI below, which also forces our 2d TQFT to be "relatively" small.



Resolving T, we obtain:



Thus, if we denote  by T_1, then

$F($  $) \cong F(\bigcirc) \otimes_{\mathbb{k}} F(T_1)$.

The cobordism map $F(S)$ is just the multiplication map. Note that the complex:

$$0 \longrightarrow F(\bigcirc) \otimes_{\mathbb{k}} F(T_1) \xrightarrow{F(S)} F(T_1) \longrightarrow 0$$

contains the subcomplex

$$0 \longrightarrow \mathbb{k} \cdot 1 \otimes_{\mathbb{k}} F(T_1) \xrightarrow{F(S)} F(T_1) \longrightarrow 0$$

which is contractible in the homotopy category. If we allow any 2d TQFT to start with, then, to obtain invariance of RI, we must require $\dim_{\Bbbk} F(\bigcirc) = 2$, so that we have the decomposition:

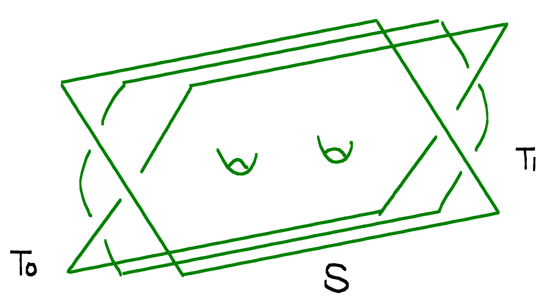
$$\begin{array}{c} 0 \longrightarrow F(\bigcirc) \otimes_{\Bbbk} F(T_1) \xrightarrow{F(S)} F(T_1) \longrightarrow 0 \\ \wr\!\wr\, S \\ 0 \longrightarrow F(T_1) \longrightarrow F(T_1) \longrightarrow 0 \\ \oplus \\ 0 \longrightarrow F(T_1) \longrightarrow 0 \end{array}$$

and in the homotopy category, we get the complex $F(T_1) \cong F(T')$.
This indeed is the case since we chose our 2d TQFT to satisfy

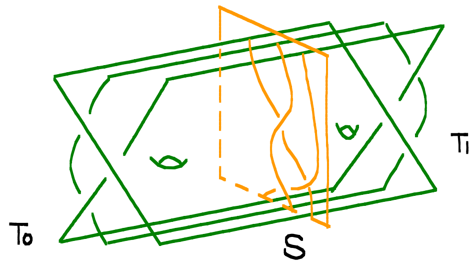
$$F(\bigcirc) = A \cong \Bbbk[X]/(X^2)$$

Rmk: In the next section we will see some variations of Khovanov homology by choosing different base rings $\Bbbk$.

For part 2) of the thm., we first exhibit how to extend F to tangle cobordisms.



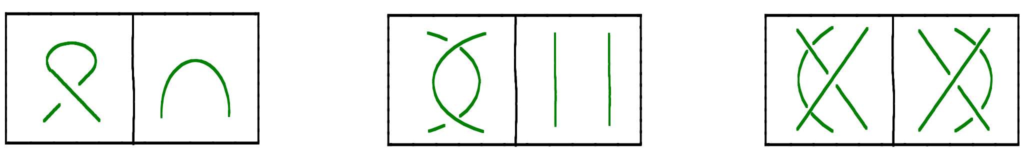
Given a tangle cobordism S between T_0 and T_1, we can chop S up by intersecting it with planes $\mathbb{R}^2 \times I \times \{t\} \subseteq \mathbb{R}^3 \times I \times I$



A section as this is called a *frame*. By chopping up fine enough, we can represent any cobordism by a sequence of frames (called a movie, not surprisingly!), such that the tangles in neighboring frames differ not too much. There are 3-types of frame changes (Roseman, Carter - Saito):

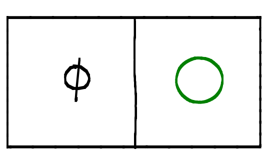
a. Isotopy moves T, H, N (there is nothing to check for them).

b. Reidemeister moves I, II, III

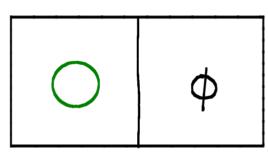


These moves do not involve the change of the topology of the tangles in the frames.

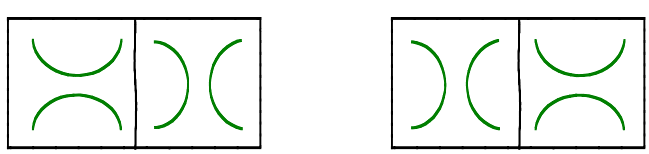
c. These are the moves that involve local changes of the topology of tangles in the frames:



Birth move



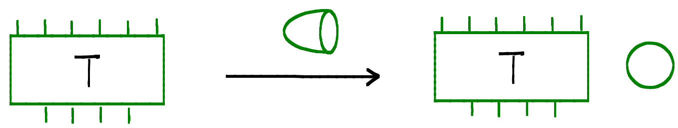
Death move



Saddle moves

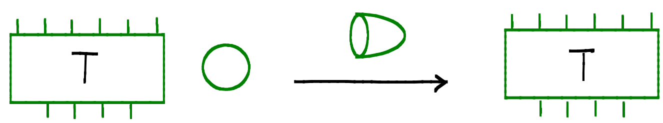
For moves of type a, b, we assign to the corresponding cobordism the isomorphism functor of complexes as in part 1).

For the birth move, we assign the inclusion:



$$\begin{array}{ccc} F(T) & \longrightarrow & F(T) \otimes_{\mathbb{k}} A \\ x & \longmapsto & x \otimes 1 \end{array}$$

For the death move, we use the trace of A:



$$\begin{array}{ccc} F(T) \otimes_{\mathbb{k}} A & \longrightarrow & F(T) \\ x \otimes a & \longmapsto & \varepsilon(a) \cdot x \end{array}$$

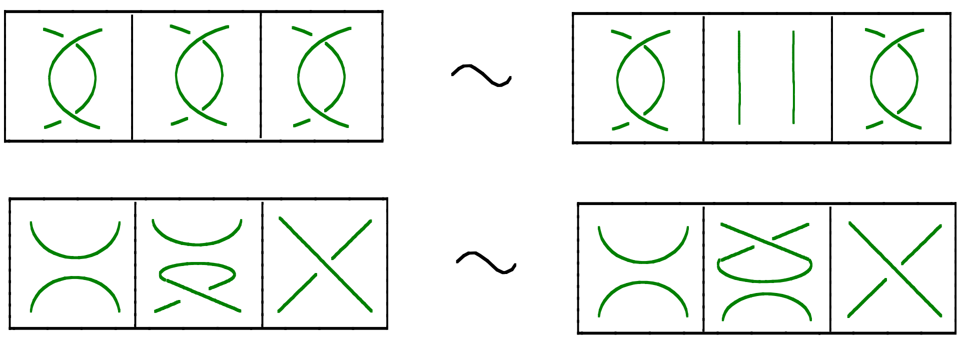
To saddle moves, we assign the map of bimodules we constructed earlier by closing up the surface with a saddle using all possible matchings and then applying our 2d TQFT.

Now part 2) of thm. 6 can be rephrased into:

> Thm 6. 2). Under this assignment, isotopic surface cobordisms give the same map of chain complexes up to a sign.

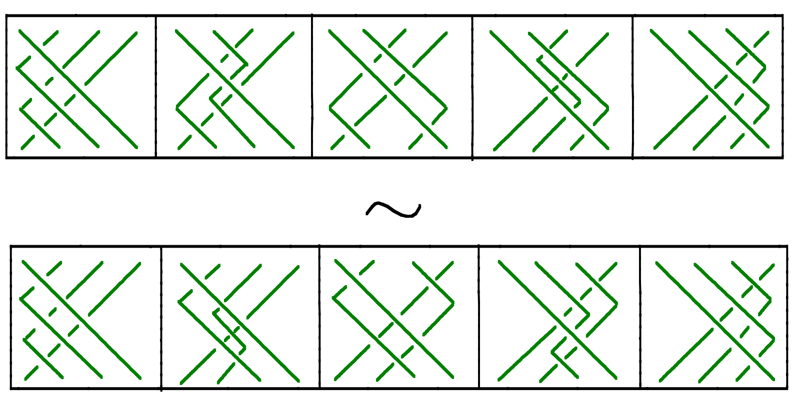
To prove it, we need to check the invariance of this assigment for isotopic cobordism surfaces when they have different movie presentations.

It's a thm. of Roseman, Carter-Saito that, different movie presentations of cobordism surfaces give rise to isotopic ones iff they can be related by a finite number of movie moves, which are classified into 31 types. Here are 2 examples:

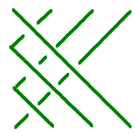
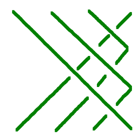


For the complete list, see their original papers.

As an example, we shall show the invariance under one of the most complicated movie moves:



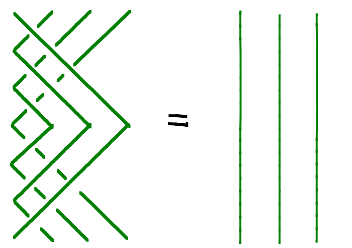
To do this, we note that

T_0  T_1 

T_0, T_1 are invertible

T_0

T_0^{-1}



Hence $F(T_0) \circ F(T_0^{-1}) = F(||||) = H^2$. This in turn implies that

$$\mathrm{Aut}(F(T_0)) \cong \mathrm{Aut}(H^2)$$

Indeed, given any automorphism α of $F(T_0)$: $\alpha: F(T_0) \longrightarrow F(T_0)$ by composing with $id: F(T_0^{-1}) \longrightarrow F(T_0^{-1})$, we get an automorphism, of H^2. Conversely, any automorphism of H^2, after composing with $id_{F(T_0)}$, gives rise to an automorphism of $F(T_0)$. One checks easily these are inverse of each other.

Lemma. $\mathrm{Aut}_{(H^n,H^n)}(H^n) = Z(H^n)_0^* \cong \{\pm 1\}$.
Pf: In general, given any ring B, then endomorphism ring of B as a (B,B)-bimodule is just the center $Z(B)$ of B. In case B is graded and we require endomorphisms preserve grading, we would get $\mathrm{End}_{(B,B)}({}_BB_B) \cong (Z(B))_0$, the deg. 0 part.
In our case of H^n, we claim that $(Z(H^n))_0 = \mathbb{Z} \cdot 1$. Thus it follows that the invertible endomorphisms can only be ± 1.

To prove the claim, we recall first that, deg 0 elements in H^n are of the form $x = \sum_{a\in B_n} x_a \cdot 1_a$, $x_a \in \mathbb{Z}$. Now $\forall\, a, b \in B_n$ two matchings, $F(W(b)a) \cong A^{\otimes k}$ for some $k \geq 1$. Choose any non-zero element $y \in F(W(b)a)$. If x is a central degree 0 element of H^n, then $xy = yx$, and

$$\begin{aligned} xy &= \textstyle\sum_{c\in B_n} x_c 1_c y = \sum_{c\in B_n} x_c \delta_{cb} y = x_b y \\ yx &= \textstyle\sum_{c\in B_n} y x_c 1_c = \sum_{c\in B_n} x_c \delta_{ac} y = x_a y \\ &\Longrightarrow \quad x_a = x_b . \end{aligned}$$

This holds for all $a, b \in B^n$. The claim follows $\square$

Now if we denote the movies in the movie move above by S and S' respectively, we get:

$$F(S) : F(T_0) \longrightarrow F(T_1)$$
$$F(S') : F(T_0) \longrightarrow F(T_1)$$

Then

$$F(S')^{-1} \circ F(S) : F(T_0) \longrightarrow F(T_0)$$
$$F(S)^{-1} \circ F(S') : F(T_1) \longrightarrow F(T_1)$$

are automorphism of $F(T_0)$, $F(T_1)$ respectively. Hence by our lemma above, we obtain:

$$F(S) = \pm F(S').$$

Rmk: It can be checked that in this case one actually has $+1$. But other movie moves give rise to a genuine -1. Works of Morrison, Clark, Walker and Caprau show that one can get rid of the sign issue at the cost of adding extra decorations to tangles and cobordism surfaces.

Rmk: Although the deg 0 part of $Z(H^n)$ is just $\mathbb{Z}$, the whole $Z(H^n)$ is more complicated. It's actually isomorphic to the cohomology ring of a certain type of Springer variety. See M. Khovanov, Crossingless Matchings and the Cohomology of (n,n) Springer Varieties.