§4. Grothendieck Groups

There are many versions of Grothendieck groups. We will need two basic versions: G_0 and K_0.

G_0

Let $\mathcal{A}$ be an (essentially small) abelian category.

Def. $G_0(\mathcal{A})$ is the abelian group generated by symbols $[M]$, where $M \in Ob(\mathcal{A})$, subject to the relations:

$$[M_2] = [M_1] + [M_3]$$

whenever

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}$.

Rmk: It follows from the definition that if $M_1 \cong M_2$ in $\mathcal{A}$, then $[M_1] = [M_2]$ in $G_0(\mathcal{A})$, and $[0] = 0$ in $G_0(\mathcal{A})$.

Recall that an object $M \in Ob(\mathcal{A})$ is said to be of finite length iff there is a finite filtration on M:

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n \supsetneq M_{n+1} = 0$$

s.t. $M_i/M_{i+1} \cong L_i$ is simple in $\mathcal{A}$.

In case M has finite length, we can easily prove by induction that:

$$[M] = \sum_{i=0}^{n} [L_i] \quad \text{in } G_0(\mathcal{A})$$

If the Jordan-Holder property holds for $\mathcal{A}$, i.e. all objects of $\mathcal{A}$ are of finite length, then it follows that $G_0(\mathcal{A})$ is isomorphic to the free abelian group generated by the isomorphism classes of simple objects:

$$G_0(\mathcal{A}) \cong \bigoplus_{i \in I} \mathbb{Z}[L_i]$$

E.g. If A is a finite dimensional $\mathbb{k}$-algebra, then finite length modules are nothing but finite dimensional modules. Thus

$$\mathcal{A} = \text{finite dim'l } A\text{-mod}$$

has the Jordan-Holder property.

For noetherian rings A, we can also consider

$$\mathcal{A} = \text{finitely generated } A\text{-mod}$$

Here finiteness is needed to avoid collapsing $G_0(\mathcal{A})$ too much:

$$A^\infty \cong A^\infty \oplus A$$

$$\Longrightarrow \quad [A^\infty] = [A^\infty] + [A]$$

$$\Longrightarrow \quad [A] = 0$$

In general, for a noetherian ring A, the categories f.ℓ. A-mod and f.g. A-mod are quite different unless A is artinian. Thus one would expect their G_0's to be quite different, although we have an obvious map:

$$G_0(\text{f.l. } A\text{-mod}) \longrightarrow G_0(\text{f.g. } A\text{-mod})$$

E.g. $A = \mathbb{Z}$.

An isomorphism class of simple $\mathbb{Z}$-modules is given by $\{\mathbb{Z}/p\mathbb{Z} \mid p: \text{prime}\}$. Thus

$$G_0(\text{f.l. } \mathbb{Z}\text{-mod}) \cong \bigoplus_p \mathbb{Z}$$

which is infinite. On the other hand, by the classification thm. of finitely generated abelian groups, any f.g. $\mathbb{Z}$-module M:

$$M \cong \mathbb{Z}^{\oplus r} \oplus \bigoplus_i \mathbb{Z}/n_i\mathbb{Z}$$

so that

$$[M] = r[\mathbb{Z}] + \textstyle\sum_i [\mathbb{Z}/n_i\mathbb{Z}]$$

But

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n_i} \mathbb{Z} \longrightarrow \mathbb{Z}/n_i\mathbb{Z} \longrightarrow 0$$

being a short exact sequence shows that

$$[\mathbb{Z}/n_i\mathbb{Z}] = [\mathbb{Z}] - [\mathbb{Z}] = 0.$$

It follows from this that $G_0(\text{f.g. } \mathbb{Z}\text{-mod}) \cong \mathbb{Z}\cdot[\mathbb{Z}]$, and the obvious map above:

$$\begin{array}{ccc} G_0(\text{f.l. } \mathbb{Z}\text{-mod}) & \longrightarrow & G_0(\text{f.g. } \mathbb{Z}\text{-mod}) \\ [\mathbb{Z}/p\mathbb{Z}] & \longmapsto & [\mathbb{Z}/p\mathbb{Z}] = 0 \end{array}$$

is the 0 map.

Ex. Compare $G_0(\text{f.l. } A\text{-mod})$ and $G_0(\text{f.g. } A\text{-mod})$ for A being $\mathbb{C}[x]$, $\mathbb{C}[x,y]$, $\mathbb{R}[x]$, and $\mathbb{C}[G]$ where G is a finite group.

If F is an exact functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$ (exact meaning that F sends short exact sequences to short exact sequences), then F descends to a homomorphism of abelian groups

$$\begin{array}{rccc} [F]: & G_0(\mathcal{A}) & \longrightarrow & G_0(\mathcal{B}) \\ & [M] & \mapsto & [F(M)] \end{array}$$

A warning is that passing to G_0 loses a lot of information about the abelian categories (basically the morphisms between objects), unless $\mathcal{A}$ is semisimple.

K_0

We shall only define K_0 for $\mathcal{A} = A$-mod, where A is a ring.

Def. $K_0(\mathcal{A})$ is the abelian group generated by symbols $[P]$, where $P \in Ob(\mathcal{A})$ is a finitely generated projective A-module, subject to the relations:

$$[P_2] = [P_1] + [P_3]$$

iff

$$P_2 \cong P_1 \oplus P_2 .$$

Here "finitely generated projective" means that P is a direct summand of some $A^{\oplus N}$ for some $N \in \mathbb{N}$, which in turn corresponds to some idempotent (projection onto P) of $End(A^{\oplus N})$. Thus to find f.g. projective modules, it's equivalent to finding idempotents of $Mat(N, End(A))$. For instance, if $e \in A$ is an idempotent: $e^2 = e$, then as a left A-module,

$${}_A A \cong {}_A Ae \oplus {}_A A(1-e)$$

so that in K_0,

$$[A] = [Ae] + [A(1-e)]$$

Similarly, if $e \in Mat(N, End A)$, then

$$N \cdot [A] = [A^{\oplus N}] = [A^{\oplus N} e] + [A^{\oplus N}(1-e)]$$

Again we stress that passing to K_0 loses too much information.

E.g. Consider a ring where $\exists\, x, y \in A$ s.t. $xy = 1$ but $yx \neq 1$. Many such rings exist in operator theory.

Then yx is an idempotent:

$$(yx)^2 = y(xy)x = yx$$

and in K_0, we have:

$$[A] = [A(yx)] + [A(1-yx)]$$

We claim that, as left A-modules,

$$A \cong Ax \cong Ayx$$

Indeed, we have the obvious inclusion map $Ayx \hookrightarrow Ax$ is also surjective, since any element ax of Ax can be rewritten as $ax = axyx = (ax)yx \in Ayx$. Thus $Ayx \cong Ax$. On the other hand

$$\begin{aligned} A &\longrightarrow Ax \\ a &\longmapsto ax \end{aligned}$$

is clearly an isomorphism of left A-modules, with inverse given by

$$\begin{aligned} Ax &\longrightarrow A \\ ax &\longmapsto axy = a. \end{aligned}$$

It follows that in $K_0(\mathcal{A})$,

$$\begin{aligned} [A] &= [Ayx] + [A(1-yx)] \\ &= [A] + [A(1-yx)] \end{aligned}$$

$$\Longrightarrow [A(1-yx)] = 0 \quad \text{in } K_0(\mathcal{A})$$

Thus passing to K_0 "forgets" the fact that $yx - 1 \neq 0$ in A.

Rmk: Construction of G_0 and K_0 generalizes without difficulty to triangulated categories, where we replace short exact sequences by distingushed triangles (with appropriate finiteness and projectivity notions). The shift functor $[1]$ desends to "$-$" in K_0:

$$[M[1]] = -[M].$$

Ring structure on K_0

In commutative algebra we know how to take tensor products of modules. In particular, $P \otimes_A -$ with a projective A-module is an exact functor, which in turn gives a ring structure on K_0, $\forall$ P, Q f.g. projective modules,

$$[P]\cdot[Q] \triangleq [P \otimes_A Q]$$

with the unit given by the symbol of A:

$$[P]\cdot[A] = [P \otimes_A A] = [P]$$

so that K_0 becomes a unital associative commutative ring.

In general, one can get $K_0(\mathcal{A})$ to be a unital associative ring whenever $\mathcal{A} = H\text{-mod}$, where H is a bialgebra. Other examples also arise from $\mathcal{D}$-modules and algebraic geometry. We will see more examples coming up later.

E.g. Topological K-theory.

For spaces that are nice enough, like finite CW complexes, we have equivalence of categories:

$$\boxed{\begin{array}{c}\text{finite dimensional}\\ \text{vector bundles}/X\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{f.g. projective}\\ C^0(X, \mathbb{R}/\mathbb{C})\text{-mod}\end{array}}$$

$$E/X \longmapsto \Gamma(X, E)$$

Thus we recover the usual topological K-theory by considering $K_0(C^0(X, \mathbb{R}))$ or $K_0(C^0(X, \mathbb{C}))$ (Swan's theorem).

It's quite remarkable how much topological information about X is remmebered by the ring $C^0(X)$, which seems to be purely algebraic.

Pairing between K_0 and G_0

We shall look at the case of $\mathcal{A} = A\text{-mod}$ where A is an artinian ring (e.g. finite dim'l $\Bbbk$-algebras). We shall see that $K_0(\mathcal{A})$ and $G_0(\mathcal{A})$ are dual to each other in some sense.

Recall the following general fact about artinian rings: Any f.d. projective module decomposes into a direct sum of finitely many indecomposable projective modules (Krull-Schmidt).

Furthermore, there exists a bijection between indecomposable projective modules and simple modules. The correspondence is given by matching a simple L_i with its projective cover P_i. We roughly sketch the proof of this fact.

Any simple left A-module L_i is of the form A/m_i where m_i is a maximal left ideal of A. Thus it's readily seen that

$$\boxed{\text{Simple } A\text{-modules}} \xleftrightarrow{1:1} \boxed{\text{Simple } A/J\text{-modules}}$$

where $J = \bigcap m_i$ is the Jacobson radical of A, which is a two-sided ideal. It's well-known that an artinian ring whose Jacobson radical is trivial is semisimple. So,

$$A/J \cong \bigoplus_{i=1}^{N} \mathrm{Mat}(n_i, D_i)$$

where D_i is a finite dimensional division algebra $/\Bbbk$. Thus the simples $\bar{L}_i$ and indecomposable projectives $\bar{P}_i$ coincide for A/J which are given by some indecomposable idempotents $\bar{e}_i \in A/J$, $i = 1, \cdots, N$. Since $J^n = 0$ for $n \gg 0$, we can lift $\bar{e}_i$ to idempotents

e_i of A via Newton's method. This in turn constructs an indecomposable projective module $P_i = Ae_i$, whose reduction mod m_i is the simple $L_i = \bar{L}_i \cong (A/J)\bar{e}_i$.

In nice enough situations, like the artinian $\mathbb{k}$-algebra case, we get a pairing between K_0 and G_0 of $\mathcal{A} = $ f.d. A-mod.

$$K_0(\mathcal{A}) \otimes_{\mathbb{Z}} G_0(\mathcal{A}) \longrightarrow \mathbb{Z}$$
$$([P], [M]) \mapsto \dim_{\mathbb{k}}(\mathrm{Hom}_A(P, M))$$

Since $K_0(\mathcal{A}) \cong \bigoplus_{i=1}^N \mathbb{Z}[P_i]$, $G_0 \cong \bigoplus_{i=1}^N \mathbb{Z}[L_i]$, the pairing is given explicitly on the bases by

$$\begin{aligned}([P_i], [L_j]) &= \dim_{\mathbb{k}} \mathrm{Hom}_A(P_i, L_j) \\ &= \dim_{\mathbb{k}} \delta_{ij} \mathrm{End}_A(L_i, L_i)\end{aligned}$$

Since L_i is simple, $\mathrm{End}_A(L_i, L_i)$ is a finite dimensional division algebra over $\mathbb{k}$. Thus $\dim_{\mathbb{k}} \mathrm{End}_A(L_i, L_i) = d_i^2$ is always a square number ($\mathrm{End}_A(L_i, L_i) \otimes_{\mathbb{k}} \bar{\mathbb{k}} \cong \mathrm{Mat}(d_i, \bar{\mathbb{k}})$ for some d_i). Thus if $\mathbb{k} = \bar{\mathbb{k}}$, $d_i = 1$, and $K_0(\mathcal{A})$ is canonically dual to $G_0(\mathcal{A})$ via this perfect pairing. This is extremely nice and we say in this case that $\mathcal{A}$ or A is absolutely irreducible.

In this artinian algebra case, we also have an obvious map of abelian groups

$$\varphi_{GK} : K_0(\mathcal{A}) \longrightarrow G_0(\mathcal{A})$$

since in this case finite length projective modules are also finite dimensional. However, this map is neither injective nor surjective. We shall see this through some examples.

E.g. $A = H^*(X, \mathbb{k})$, where X is a connected CW complex.

In this case A is a graded local ring whose Jacobson radical equals the unique maximal ideal which consists of elements of strictly positive degree : $m = H^{>0}(X, \Bbbk)$. Thus A has a unique simple module $\Bbbk \cong A/m$, with all elements in m acting trivially on it. Thus

$$G_0(\mathcal{A}) = \mathbb{Z}[\Bbbk].$$

On the other hand, any projective module over a local ring is isomorphic to a direct sum of free modules (Nakayama's lemma), so

$$K_0(\mathcal{A}) = \mathbb{Z}[A].$$

Thus

$$\begin{aligned} \varphi_{GK} : K_0(\mathcal{A}) &\longrightarrow G_0(\mathcal{A}) \\ [A] &\mapsto [A] = \dim_{\Bbbk}(A) \cdot [\Bbbk]. \end{aligned}$$

This won't be surjective unless $A \cong \Bbbk$.

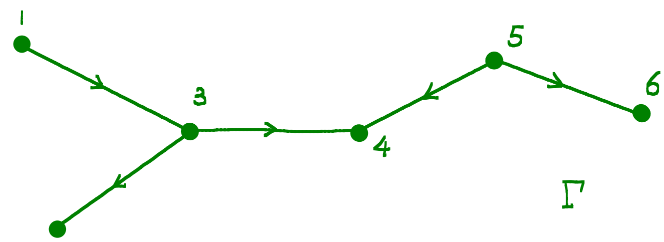
We shall see more examples coming up in the next subsection when we discuss path algebras and nil-Coxeter algebras.

Example: path algebras and their Grothendieck groups

One way of constructing many interesting finite dimensional algebras is through path algebras. We first recall their definitions and basic properties.

Let Γ be an oriented graph and $\Bbbk[\Gamma]$ the $\Bbbk$-vector space with a basis spanned by all oriented path in Γ (vertices are counted as length 0 paths). The product structure on $\Bbbk[\Gamma]$ is given by concantenation of paths.

E.g.



In the above example, Γ has as a basis the paths of:

length 0: (1), (2), (3), (4), (5), (6)

length 1: (13), (34), (54), (56), (32)

length 2: (134), (132)

with products:

$$(1)\cdot(13) = (13), \quad (13)\cdot(1) = 0, \quad (134)(1) = 0, \quad (13)(34) = (134), \quad (34)(13) = 0, \text{ etc.}$$

Path algebras enjoy the following nice properties:

- $\mathbb{k}[\Gamma]$ is a unital, associative algebra. It's finite dimensional iff there is no oriented loop in Γ.

- Their homological dimension is 1, i.e. any submodule of any projective module is projective.

- The Jacobson radical $J(\mathbb{k}[\Gamma])$ is spanned by all paths of length ≥ 1. $\mathbb{k}[\Gamma]/J \cong \prod_{i \in \mathrm{vert}(\Gamma)} \mathbb{k}(i)$, which is semisimple.

- The complete set of indecomposable idempotents is given by the set of vertices:

$$1 = \sum_{i \in \mathrm{vert}(\Gamma)} (i)$$

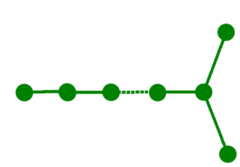
- Mod-$\mathbb{k}[\Gamma]$. For our purpose it's more convenient to consider right $\Bbbk[\Gamma]$-modules. Any finite dimensional right $\Bbbk[\Gamma]$-module can be decomposed by the set of idempotents:

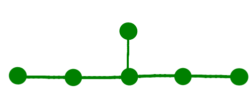
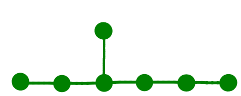
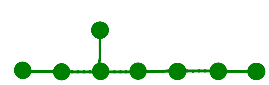
$$M \cong \bigoplus_{i \in \mathrm{vert}(\Gamma)} M(i)$$

and the edges (ij) now become linear maps between vector spaces $M(i) \xrightarrow{(ij)} M(j)$. Conversely, any such datum $\{(M_i, \varphi_{ij}) \mid (i) \in \mathrm{vert}(\Gamma),\ (ij) \in \mathrm{edge}(\Gamma)\}$ defines a right $\Bbbk[\Gamma]$-module.

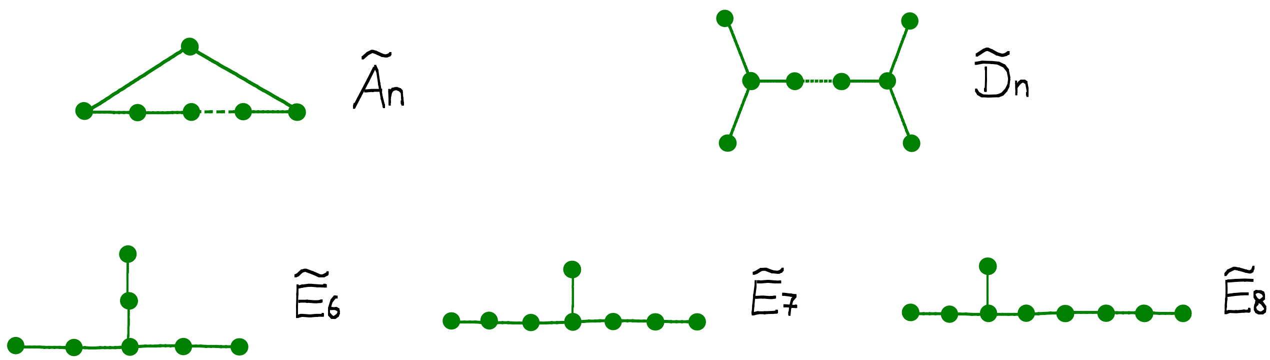
As in the general theme of representation theory, whenever one has an interesting algebra in mind ($\Bbbk[G]$, $\mathcal{U}(\mathfrak{g})$ etc.), one tries to classify all its representations. This is in general too hard, so one restricts to some smaller category of representations (e.g. finite dimensional, integrable, highest weight etc.). Then one finds that for nice enough cases (e.g. $\mathbb{C}[G]$, $|G|<\infty$, $\mathcal{U}(\mathfrak{g})$, $\mathfrak{g}$: semisimple Lie algebra) one gets a complete list of representations. These were the cases when A is semisimple (homological dimension 0). The next step would be to move on to homological dimension one cases (e.g. $\mathcal{O}_F$, F: number field; algebraic curves; $\Bbbk[\Gamma]$ etc.). In this respect, one has:

Thm. (1). $\Bbbk[\Gamma]$ has finitely many non-isomorphic indecomposable representations iff Γ has its underlying graph of Dynkin type (A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7, E_8). In this case the indecomposables are in bijection with positive roots of the corresponding Lie algebra.

 : A_n  : D_n

 : E_6  : E_7  : E_8

(2). $\Bbbk[\Gamma]$ has 1-parameter family of non-isomorphic indecomposable modules for each dimension vector (i.e. a non-negative element in $G_0(\Bbbk[\Gamma])$) iff Γ has its underlying graph of affine type: $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$:



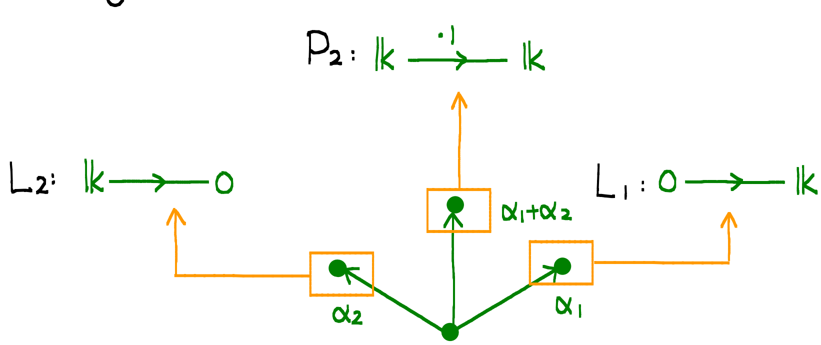
(3). For any other Γ, there are higher dimensional moduli spaces parametrizing non-isomorphic indecomposable module.

Case (1) is said to be of finite rep'n type, while case (2) & (3) are said to be of tame and wild rep'n types respectively.

E.g. We look at one of the easiest examples: A_2.

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet}$$

There are 3 indecomposable right $\Bbbk[\Gamma]$-modules, corresponding to positive roots of $\mathfrak{sl}_3$:



We observe that:

- The module L_1 is simple and projective, since $L_1 = (1)\cdot \Bbbk[A_2]$
- The module L_2 is simple but not projective.
- The module $P_2 = (2)\cdot \Bbbk[A_2]$ is projective. It's an non-trivial extension of L_2 by L_1:

$$0 \longrightarrow L_1 \longrightarrow P_2 \longrightarrow L_2 \longrightarrow 0$$

which, written out as vector spaces with maps, corresponds to the commutative diagram (note that reversing the vertical arrows doesn't work!):

$$\begin{array}{ccccc}
\Bbbk & \longrightarrow & 0 & : & L_2 \\
\uparrow{\scriptstyle \cdot 1} & & \uparrow & & \\
\Bbbk & \longrightarrow & \Bbbk & : & P_2 \\
\uparrow & & \uparrow{\scriptstyle \cdot 1} & & \\
0 & \longrightarrow & \Bbbk & : & L_1
\end{array}$$

Thus on the Grothendieck groups, we have:

$$K_0(\Bbbk[A_2]) \cong \mathbb{Z}[L_1] \oplus \mathbb{Z}[P_2]$$
$$G_0(\Bbbk[A_2]) \cong \mathbb{Z}[L_1] \oplus \mathbb{Z}[L_2]$$

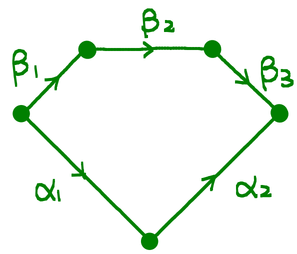
and φ_{GK} is of the form:

$$\begin{aligned}
\varphi_{GK} : K_0(\Bbbk[A_2]) &\longrightarrow G_0(\Bbbk[A_2]) \\
[L_1] &\longmapsto [L_1] \\
[P_2] &\longmapsto [L_1] + [L_2]
\end{aligned}$$

Example: quotient path algebras

The path algebra construction of artinian algebras has a variation, namely we can mod out relations among paths.

Notice that, if the relations we are modding out lie inside $J(A)$, it doesn't change the sizes of Grothendieck groups K_0 and G_0.

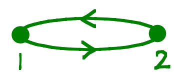


Γ

$$A \triangleq \Bbbk[\Gamma]/\langle \alpha_1\alpha_2 + \lambda\beta_1\beta_2\beta_3 \rangle$$
$$\lambda \in \Bbbk$$

E.g. Quotients of path algebras:

(1). Consider the quotient path algebra $A \triangleq \Bbbk[\Gamma]/\langle (121)=(1),\ (212)=(2)\rangle$



Γ

What we get is a 4-dim'l algebra spanned by the paths $(1), (2), (12), (21)$. It's readily seen that this algebra is isomorphic to the 2×2 matrix algebra $M(2, \Bbbk)$:

$$(1) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (2) \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (12) \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (21) \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The relations do not lie in $J(A)$ and the sizes of K_0 and G_0 change:

$$K_0(A) \cong \mathbb{Z} \cong G_0(A)$$

and $\varphi_{GK} = 1$.

(2). We consider the same Γ but different relations.

$$A \triangleq \Bbbk[\Gamma]/\langle (121), (212)\rangle$$

A is still 4-dim'l but the sizes of K_0 and G_0 are different from above. They are both of rank 2 now.

Simples: $L_1: \Bbbk \rightleftarrows 0$, $L_2: 0 \rightleftarrows \Bbbk$

Projectives: $P_1 : \Bbbk \underset{\cdot 1}{\overset{0}{\rightleftarrows}} \Bbbk \qquad P_2 : \Bbbk \underset{\cdot 0}{\overset{\cdot 1}{\rightleftarrows}} \Bbbk$

Thus we have extension relations:

$$0 \longrightarrow L_2 \longrightarrow P_1 \longrightarrow L_1 \longrightarrow 0$$
$$0 \longrightarrow L_1 \longrightarrow P_2 \longrightarrow L_2 \longrightarrow 0 .$$

This in turn implies that the natural map φ_{GK} is neither injective nor surjective.

$$\begin{aligned} \varphi_{GK} : K_0(\mathcal{A}) &\longrightarrow G_0(\mathcal{A}) \\ [P_1] &\longmapsto [L_1]+[L_2] \\ [P_2] &\longmapsto [L_1]+[L_2] \end{aligned}$$

Now we have seen that φ_{GK} is in general neither injective nor surjective. Nevertheless, we have a necessary condition when A is finite dimensional.

Prop. If A is finite dimensional and has finite homological dimension, then:

$$\varphi_{GK} : K_0(\mathcal{A}) \longrightarrow G_0(\mathcal{A})$$

is an isomorphism.

Pf: If so, every simple module has a finite step projective resolution:

$$0 \longrightarrow P_{i_k} \longrightarrow \cdots \longrightarrow P_{i_1} \longrightarrow L_i \longrightarrow 0 .$$

Then, in $G_0(\mathcal{A})$, we have,

$$[L_i] = \textstyle\sum_{\alpha=0}^{k} (-1)^{\alpha} [P_{i_\alpha}]$$

Thus $[L_i] \in \operatorname{Im} \varphi_{GK}$. But since A is finite dimensional, $K_0(\mathcal{A})$ and $G_0(\mathcal{A})$ are free abelian groups of the same rank. Thus φ_{GK} must be an isomorphism, since it's an epimorphism of free $\mathbb{Z}$-modules of the same rank. $\square$

Example: nil-Coxeter rings

Now we shall give an interesting example of functors inducing maps on Grothendieck groups.

Recall that if $F: \mathcal{A} \longrightarrow \mathcal{B}$ is an exact functor between abelian groups, it desends to abelian group homomorphisms on Grothendieck groups:

$$[F]: G_0(\mathcal{A}) \longrightarrow G_0(\mathcal{B})$$
$$[F]: K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{B})$$

But in practice, how does one get exact functors between abelian categories, say, the most common cases $\mathcal{A} = A\text{-mod}$, $\mathcal{B} = B\text{-mod}$?

One could try tensoring A-modules with a (B, A)-bimodule ${}_BN_A$:

$${}_BN_A \otimes_A - \; : \; A\text{-mod} \longrightarrow B\text{-mod}.$$

However, this is in general only right exact:

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$
$$\Longrightarrow \quad {}_BN_A \otimes_A M \longrightarrow {}_BN_A \otimes_A M \longrightarrow {}_BN_A \otimes_A M \longrightarrow 0$$

To restore exactness, one can require N to be right A-projective or more generally right A-flat. Moreover, to get maps on K_0's, we need ${}_BN_A \otimes_A -$ to send projective A-modules to projective B-modules. For this it suffices to take N to be left B-projective as well, since if P is any projective A-module, then $P \oplus Q \cong A^n$ for some A-module Q, and

$$\begin{aligned} {}_BN_A \otimes_A P \oplus {}_BN_A \otimes_A Q &\cong {}_BN_A \otimes_A (P \oplus Q) \\ &\cong {}_BN_A \otimes_A A^n \\ &\cong N^n, \end{aligned}$$

which implies that ${}_BN_A \otimes_A P$ is projective.

Henceforth we shall always take the (B, A)-bimodule N to be both left and right projective. We shall also denote the map induced by N on K_0 and G_0 by $[N]$.

Def. (Nil-Coxeter ring on n strands)
The nil-Coxeter ring on n strands NC_n is the $\Bbbk$-algebra generated by generators T_i, $i = 1, \cdots, n-1$, subject to relations:

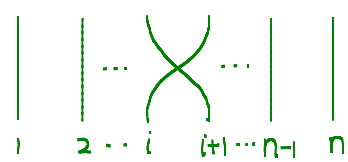
$$\text{(i)}. \quad T_i^2 = 0$$

$$\text{(ii)}. \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

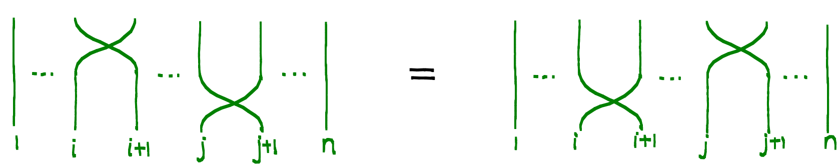
$$\text{(iii)}. \quad T_i T_j = T_j T_i \qquad |i-j| > 1$$

It admits the following graphical presentation, where T_i is depicted by a crossing:

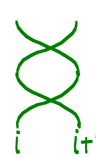
T_i =



The condition (iii) becomes a planar isotopy relation of far away strands:

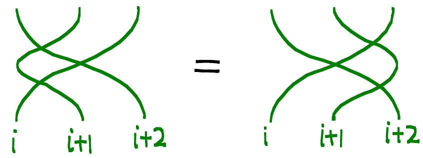


so that all relations can be drawn locally. Relation (i) says that

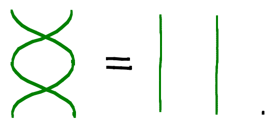


$= 0$,

while relation (ii) is a Reidemeister III move:



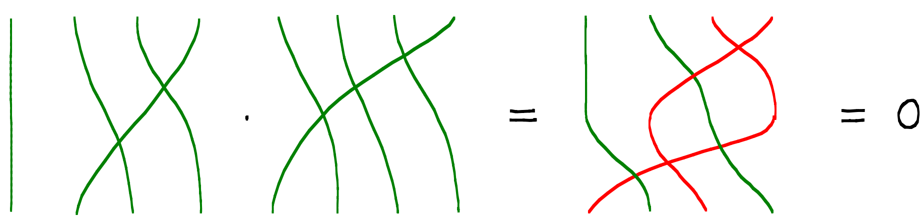
Rmk: NC_n is a deformed version of $\Bbbk[S_n]$ whese $s_i^2 = 1$:



One can show as for $\Bbbk[S_n]$ that:

- NC_n is a graded local ring by setting $\deg T_i = 1$, and
$$J(NC_n) = (T_1, \cdots, T_{n-1})$$
- It's a finite dimensional $\Bbbk$-algebra of dimension $n!$. A basis of NC_n is parametrized by S_n: fix a reduced expression for each element $w \in S_n$: $w = s_{i_1} \cdots s_{i_r}$. Then we set
$$T_w \triangleq T_{i_1} \cdots T_{i_r}$$
- The ring structure of NC_n is given by:
$$T_{w_1} \cdot T_{w_2} = \begin{cases} T_{w_1 \cdot w_2} & \text{if } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2) \\ 0 & \text{if } \ell(w_1 w_2) > \ell(w_1) + \ell(w_2) \end{cases}$$

i.e. the product is 0 whenever two strands cross each other inside the picture:



It follows that f.d. NC_n-mod has a unique irreducible module $\Bbbk \cong NC_n / (T_1, \cdots, T_{n-1})$, and a unique projective module NC_n.

In $G_0(NC_n)$, we have:

$$[NC_n] = n!\,[\Bbbk]$$

- There are induction functors

$$\mathrm{Ind}_n : NC_n\text{-mod} \longrightarrow NC_{n+1}\text{-mod}$$

coming from the obvious inclusion $NC_n \hookrightarrow NC_{n+1}$ (pictorially just put one more straight strand on the r.h.s. of any picture in NC_n), i.e.

$$\mathrm{Ind}_n = {}_{NC_{n+1}}NC_{n+1} \otimes_{NC_n} - : NC_n\text{-mod} \longrightarrow NC_{n+1}\text{-mod}.$$

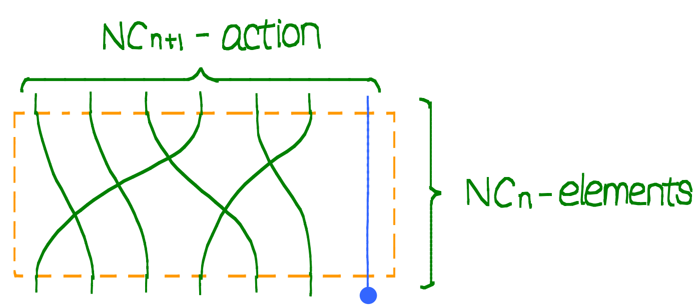
It's left adjoint to the restriction functor:

$$\mathrm{Res}_{n+1} = {}_{NC_n}NC_{n+1} \otimes_{NC_{n+1}} - : NC_{n+1}\text{-mod} \longrightarrow NC_n\text{-mod}.$$

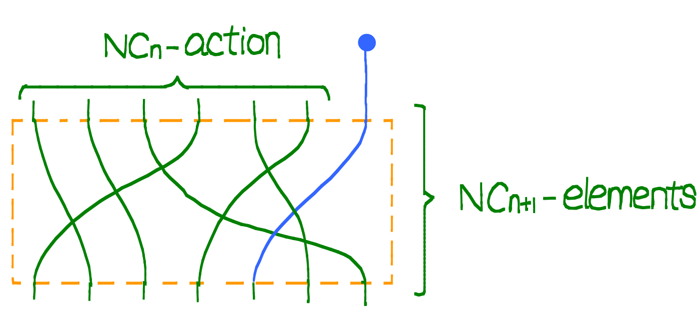
(We shall denote for short ${}_{NC_n}NC_{n+1\,NC_{n+1}}$ by ${}_n(NC_{n+1})_{n+1}$, and $\otimes_{NC_n}$ by $\underset{n}{\otimes}$).

Diagramatically, we can depict these functors as follows (where we turn out head 90° and read pictures from bottom up).

Ind_n :



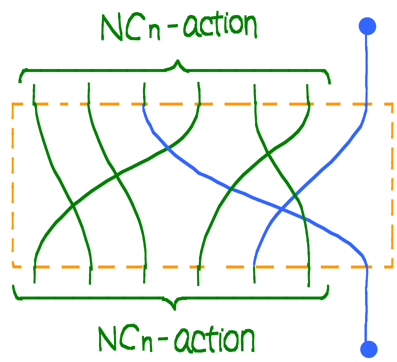
Res_{n+1} :



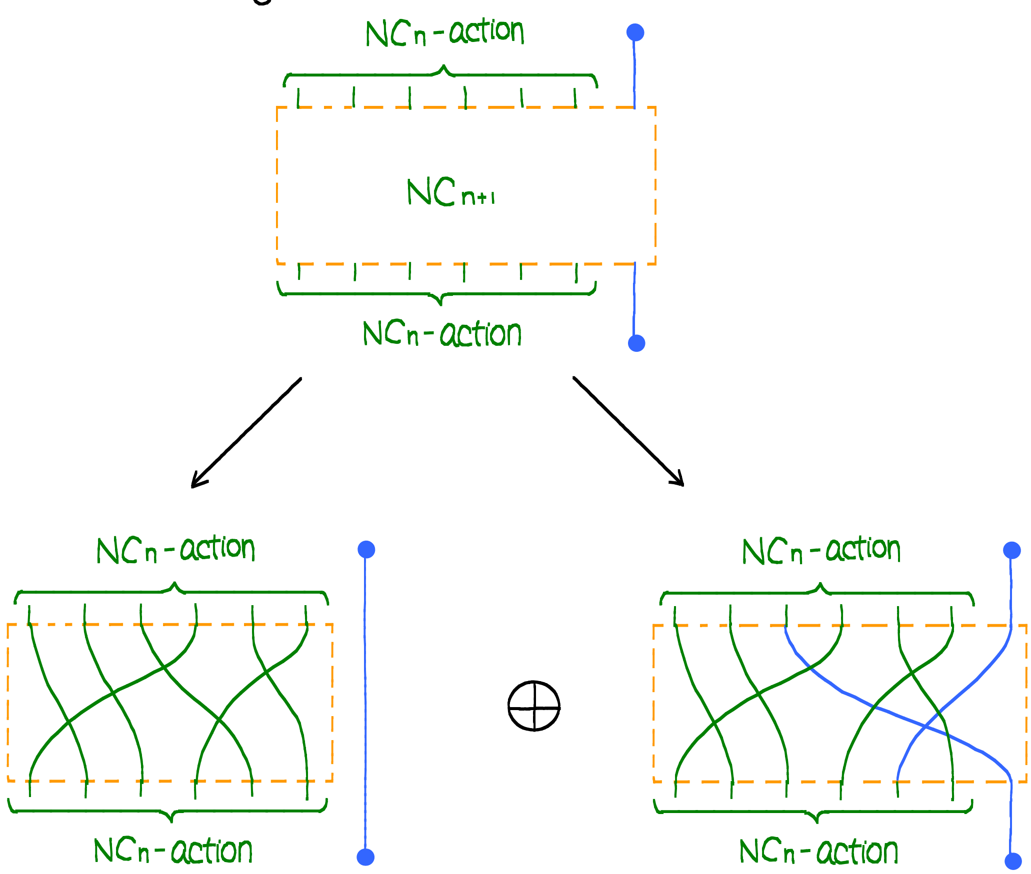
The composition $NC_n\text{-mod} \longrightarrow NC_n\text{-mod}$:

$$\mathrm{Res}_{n+1} \circ \mathrm{Ind}_n = {}_n(NC_{n+1}) \underset{n+1}{\otimes} {}_{n+1}(NC_{n+1})_n \underset{n}{\otimes} -$$
$$= {}_n(NC_{n+1})_n \underset{n}{\otimes} -$$

Let's look at the (NC_n, NC_n)-bimodule ${}_n(NC_{n+1})_n$ more closely.
The bimodule consists of pictures:

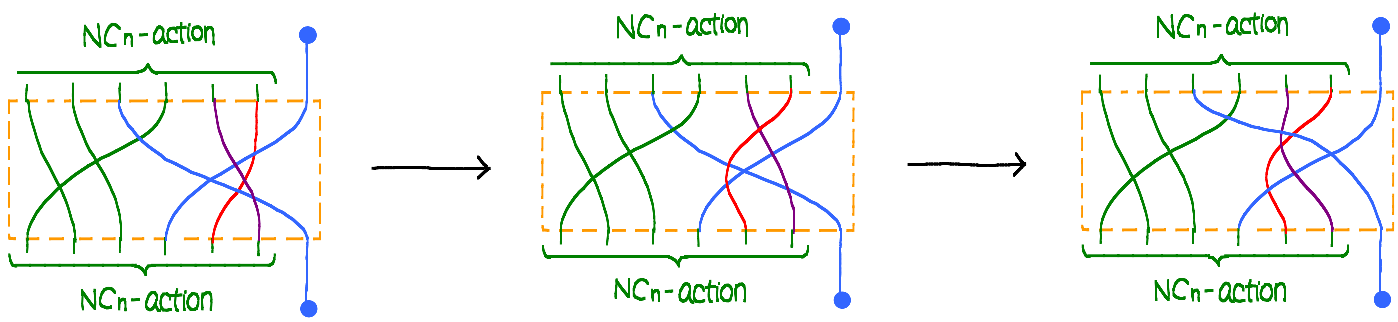


This bimodule naturally splits into two summands:

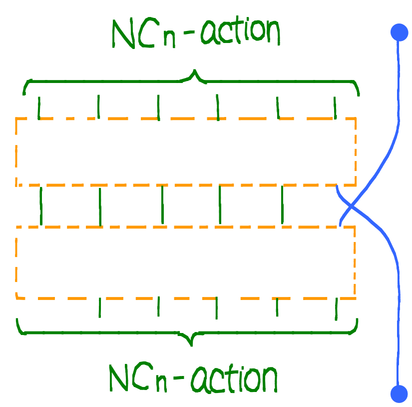


The first pictures constitute a copy of ${}_n(NC_n)_n$. We analyse the second summand.

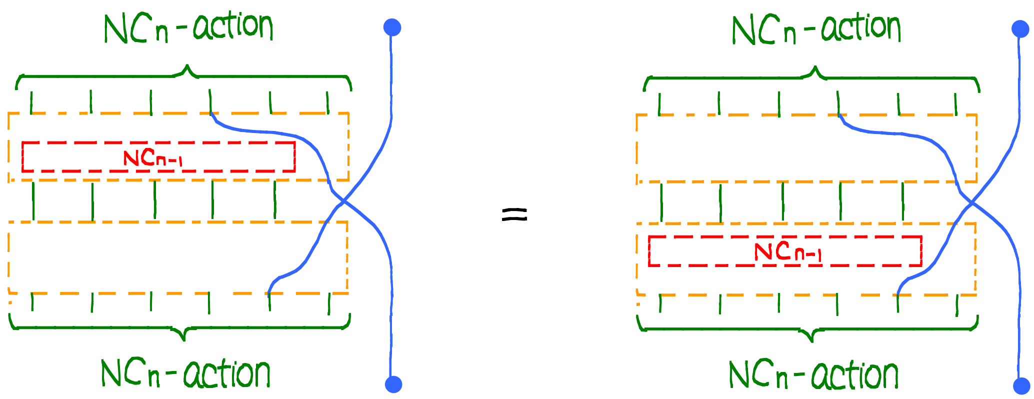
The second summand consists of pictures where the blue strands always cross each other. Making sufficiently many RIII moves, we may arrange the blue crosses appear at the right end of the picture:



Thus by putting the blue cross in the right end and middle of the box, we may always arrange pictures to look like:



Now we readily see that any element from NC_{n-1} can pass freely from the bottom box to the top box, and vice versa:



Hence we can conclude that the second summand is nothing but

$$(NC_n \underset{n-1}{\otimes} NC_n)$$

and what it does when tensored with an NC_n-module is just

$$(NC_n \underset{n-1}{\otimes} NC_n) \underset{n}{\otimes} M = Ind_{n-1} \circ Res_n (M).$$

In conclusion, we have established the isomorphism of functors:

$$\boxed{Res_{n+1} \circ Ind_n \cong Id_n \oplus Ind_{n-1} \circ Res_n}$$

Now we look at what this functor equation does on K_0 and G_0. i.e. the "decategorification" of this equation.

It will be convenient for us to take all n into account, since Ind, Res change categories 1 by 1. Thus we define:

$$NC \triangleq \bigoplus_{n\geq 0} NC_n$$
$$X \triangleq \bigoplus_{n\geq 0} Ind_n$$
$$D \triangleq \bigoplus_{n\geq 0} Res_{n+1}$$

Then:

$$K_0(NC) \cong \bigoplus_{n\geq 0} K_0(NC_n)$$
$$\cong \bigoplus_{n\geq 0} \mathbb{Z}[NC_n]$$

We will formally write $[NC_n]$ as x^n. since

$$NC_n = \mathrm{Ind}_{n-1}(NC_{n-1})$$
$$= \mathrm{Ind}_{n-1} \circ \mathrm{Ind}_{n-2}(NC_{n-2})$$
$$= \cdots\cdots$$
$$= \mathrm{Ind}_{n-1} \circ \cdots \circ \mathrm{Ind}_0(\Bbbk)$$

is the repeated induction (our X functor) of the trivial module of NC_0. Thus we identify $K_0(NC) \cong \mathbb{Z}[x]$.

Similarly.

$$G_0(NC) \cong \bigoplus_{n\geq 0} G_0(NC_n)$$
$$\cong \bigoplus_{n\geq 0} \mathbb{Z}[NC_n/J(NC_n)]$$

But in G_0, $[NC_n] = n!\,[NC_n/J(NC_n)]$. Thus we again formally write

$$[NC_n/J(NC_n)] = \frac{1}{n!}[NC_n]$$
$$= \frac{1}{n!}x^n$$

Then in this notation

$$[X]: K_0(NC) \longrightarrow K_0(NC)$$
$$x^n \longmapsto x^{n+1}$$
$$[X]: G_0(NC) \longrightarrow G_0(NC)$$
$$\frac{x^n}{n!} \longmapsto \frac{x^{n+1}}{(n+1)!}\cdot(n+1)$$

i.e. $[X]$ is just the multiplication by x on Grothendieck groups.

On the other hand.

$$[D]: K_0(NC) \longrightarrow K_0(NC)$$
$$x^{n+1} \longmapsto (n+1)x^n$$
$$G_0(NC) \longrightarrow G_0(NC)$$
$$\frac{x^n}{n!} \longmapsto \frac{x^{n+1}}{(n+1)!}$$

Hence $[\mathcal{D}]$ acting on the Grothendieck groups $K_0(NC) \cong \mathbb{Z}[x]$, $G_0(NC) \cong \bigoplus_{n\geq 0} \mathbb{Z}[\frac{x^n}{n!}] \subseteq \mathbb{Z}[x]$ is just the usual differential operator.

In sum, we have established:

Thm. (Nil-Coxeter ring categories the first Weyl algebra, part I).
The Nil-Coxeter rings together with the induction, restriction functors $\mathcal{X}$ and $\mathcal{D}$ categorify the polynomial representation of the first Weyl algebra $\mathbb{Z}\langle x, \partial\rangle/(\partial x - x\partial - 1)$ □

Moreover, as we mentioned before, the pairing between K_0 and G_0 is just the "decategorification" of the Hom space:

$$(\ , \ ): K_0(NC_n) \otimes_{\mathbb{Z}} G_0(NC_n) \longrightarrow \mathbb{Z}$$
$$([P] \ , \ [M]) \mapsto \dim_{\mathbb{k}} \mathrm{Hom}_{NC_n}(P, Q)$$

Thus on $K_0(NC)$ and $G_0(NC)$, we have

$$\left(x^n, \frac{x^m}{m!} \right) = \dim_{\mathbb{k}} \mathrm{Hom}_{NC}(NC_n, NC_m/J(NC_m)) = \delta_{n,m},$$

which in turn implies that, if we introduce the bilinear form

$$(x^n, x^m) = \delta_{n,m}\, n!$$

then x, ∂_x become adjoint operators under this inner product:

$$\begin{aligned}(x\cdot x^n, x^m) &= \delta_{n+1,m}\,(n+1)! \\ &= \delta_{n,m-1}\, m\cdot n! \\ &= (x^n, \partial_x(x^m)).\end{aligned}$$

Thm. (Nil-Coxeter ring categories the first Weyl algebra, part II).
The adjoint functors $\mathcal{X}/\mathcal{D}$ and Hom_{NC} categorify the adjoint

operators x, ∂_x and the bilinear pairing $(\ ,\ )$. $\square$

Rmk: This story can be pushed further. If we use the obvious inclusion $NC_n \times NC_m \hookrightarrow NC_{n+m}$ and extend it to $NC \times NC \longrightarrow NC$, this would give us a categorification of $\mathbb{Z}[x]$ as a bialgebra, and even moreover, as a Hopf algebra.