Categorification

Note Title 9/7/2010

Lecture notes for a topic course
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Columbia University. Fall 2010
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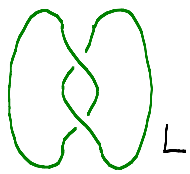
§1. Introduction

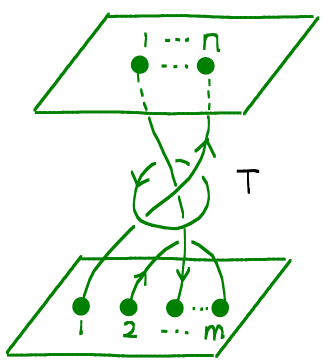
Quantum knot invariants

It's well-known from the work of Jones, Witten, Reshetikhin – Turaev etc. how to construct quantum invariants of links $P(L) \in \mathbb{Z}[q,q^{-1}]$:



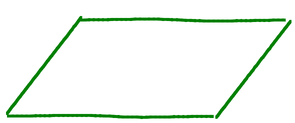
$$\longmapsto \quad P_L \in \mathbb{Z}[q,q^{-1}]$$

One breaks knots / links into composition of tangles, and associate to each tangle a linear map between certain vector spaces:



$$\begin{array}{ccc} & \longmapsto & V(n) \\ & & \uparrow F_T \\ & \longmapsto & V(m) \end{array}$$

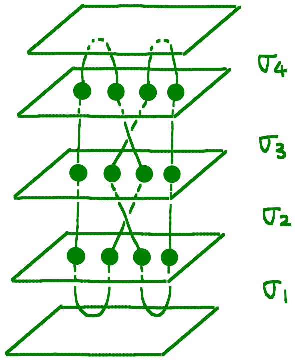
and to the empty plane with no points one associates some base ring A (e.g. $\mathbb{Q}(q)$ or $\mathbb{Z}[q,q^{-1}]$).



$$\longmapsto \quad A$$

Then a link determines an endomorphism of A, or equivalently, an element $f \in A$.

Very often, the vector spaces $V(n)$'s carry extra structures. For instance, they might be modules of quantum groups $U_q(\mathfrak{g})$.

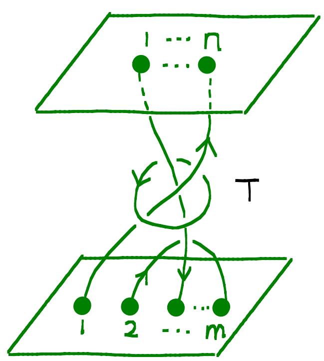


$$\longmapsto \quad \left.\begin{array}{c} A \\ \uparrow F_{\sigma_4} \\ V(4) \\ \uparrow F_{\sigma_3} \\ V(4) \\ \uparrow F_{\sigma_2} \\ V(4) \\ \uparrow F_{\sigma_1} \\ A \end{array}\right\} f = F_{\sigma_4}\circ F_{\sigma_3}\circ F_{\sigma_2}\circ F_{\sigma_1} \in A$$

Categorifying knot invariants

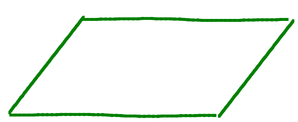
The quantum tangle invariants are usually integral and one can understand this by categorifying the above picture.

To each plane with n points we assign a category $\mathcal{C}(n)$, e.g. modules over some ring R_n : R_n-mod, and to each tangle T we assign a functor $F_T : \mathcal{C}(\partial_0 T) \longrightarrow \mathcal{C}(\partial_1 T)$, e.g. tensoring with an (R_m, R_n)-bimodule $({}_{R_m}M_{R_n} \otimes -) : R_n\text{-mod} \longrightarrow R_m\text{-mod}$



$$\begin{array}{cc} \longmapsto & \mathcal{C}(n) \\ & \uparrow F_T \\ \longmapsto & \mathcal{C}(m) \end{array}$$

and to empty planes we assign some base category. for instance the category of graded vector spaces.

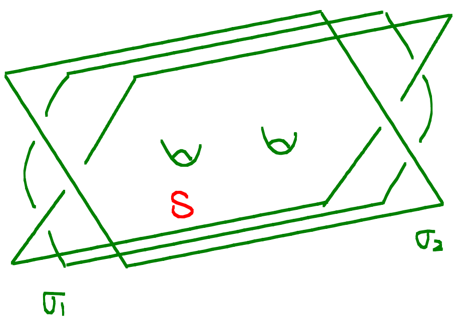


$$\longmapsto \quad \mathcal{C}_0$$

Then what one gets for a link L is ideally a functor by tensoring with a graded vector space M_L, whose graded dimension recovers the usual quantum knot invariants:

$$L \longmapsto M_L \longmapsto \mathrm{gr.dim}\, M_L = \sum_{n \in \mathbb{Z}} \dim M_L(n)\, q^n = P_L$$

What one gains from this complicated process is the functoriality, namely, the assignments above extend naturally to cobordisms of tangles, in many cases

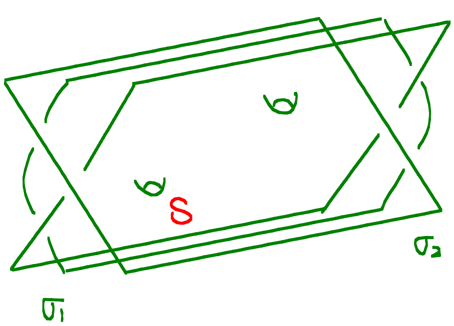


A cobordism S between two tangles σ_1 and σ_2

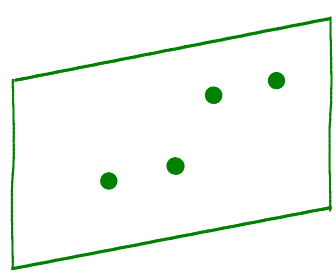
$\longmapsto$ Natural transformation between the functors F_{σ_1} and F_{σ_2}:

$F_S : F_{\sigma_1} \Longrightarrow F_{\sigma_2}$

Here $S \subseteq \mathbb{R}^2 \times [0,1] \times [0,1]$ is a surface with boundaries and corners, and one requires further that the projection of S onto $I \times I$ be ramified with double branch points only.



$\longmapsto$

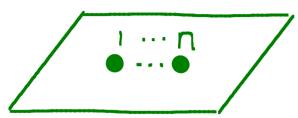


The double branch points correspond to moves

The algebraic gadget F_S is an algebraic invariant associated with the 4 dim'l object $(S, \mathbb{R}^2 \times I^2)$, which should be sensitive to the smooth structures of $\mathbb{R}^4$.

Triangulated categories

The previous discussion is too ideal if we insist on using abelian categories: in most cases, abelian categories are too rigid to carry interesting symmetries. In the simplest example, if the abelian categories involved are semisimple,



$\longmapsto$ A semisimple abelian category $\mathcal{A}$

and since the braid group on n strands Br_n acts on the left, it acts as automorphisms of $\mathcal{A}$ as well. Since $\mathcal{A}$ is semisimple, it doesn't have that many interesting automorphisms other than permuting the simple objects (being simple is a categorical notion that should be preserved under automorphisms of abelian categories).

However, as we will later see, things get more interesting when we pass to triangulated categories. Recall that in homological algebra, one can obtain triangulated categories from abelian categories:

$\mathcal{A}$: abelian category, e.g. $\mathcal{A}$ = A-mod	$\longrightarrow$	$\mathrm{Comp}(\mathcal{A})$: complexes of objects of $\mathcal{A}$. (abelian)	$\longrightarrow$	$C(\mathcal{A})$: complexes of objects of $\mathcal{A}$ up to chain-homotopy (triangulated)

Forming complexes

Factoring out the ideal of null-homotopic chain maps

$\longrightarrow$ $\boxed{\begin{array}{c}D(\mathcal{A})\text{: the derived} \\ \text{category of } \mathcal{A} \\ \text{(triangulated)}\end{array}}$

$\underbrace{\qquad\qquad\qquad}_{\text{Inverting quasi-isomorphisms}}$

In the realm of triangulated categories, say, $C(\mathcal{A})$, one obtains many more interesting endomorphisms given by tensoring with complexes of projective bimodules (or derived tensor if working with $D(\mathcal{A})$).

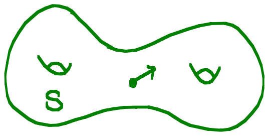
Interesting examples of triangulated categories arise from modular representation theory, algebraic geometry, symplectic geometry etc.

E.g. In modular representation theory of finite groups G over an algebraically closed field $\mathbb{k}$ of char $p \mid |G|$, the group algebra $\mathbb{k}G$ decomposes into indecomposable blocks:

$$\mathbb{k}G \cong B_1 \times \cdots \times B_n.$$

Some of the blocks are matrix algebras $\mathrm{Mat}(n_i, \mathbb{k})$, which are easy to deal with, while the other "interesting" blocks may sometimes have faithful braid group actions on their derived categories! We will see some examples later.

E.g. In Heegaard-Floer theory, one associates to a compact surface with a fixed tangent vector on it a derived category of certain dg-algebras. Unlike the local nature of a plane with n dots, the category D^b(some dg-algebra) already encodes some global topological information.



$\longmapsto$ D^b (Lipshitz-Ozsvath-Thurston dg-algebra)

E.g. In geometric representation theory, people used to study various varieties arising from representation theory. One takes the triangulated categories of complexes of sheaves (coherent or constructible) as the underlying categories. Then one can usually realize various algebra actions on the homology or K-groups of some collection of varieties, with the generators of algebras acting via correspondences between varieties ("de-categorification" of the categories above).

Recently people started to realize the importance of studying these categories themselves and their 2-morphisms.

Quantum groups

As mentioned in the beginning, we will be dealing with quantum groups and their representations. Here we briefly explain why they are interesting to us.

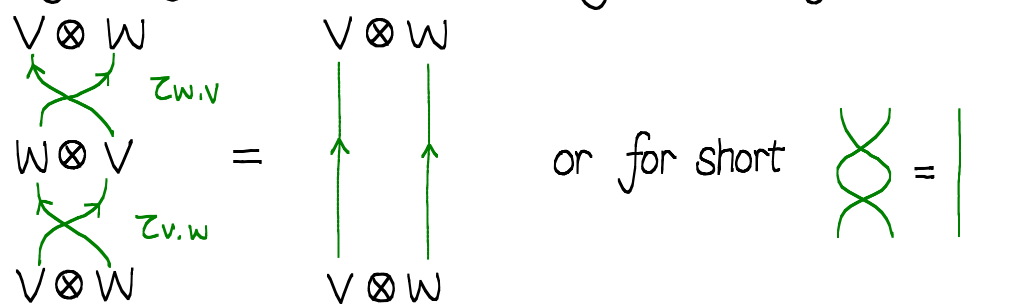
Recall that for any Lie algebra $\mathfrak{g}$, we can associate with it a Hopf algebra $U(\mathfrak{g})$ such that the representation category of $\mathfrak{g}$ is isomorphic to the module category of $U(\mathfrak{g})$. Moreover, the comultiplication:

$$\begin{aligned} \Delta : U(\mathfrak{g}) &\longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g}) \\ x &\longmapsto x \otimes 1 + 1 \otimes x \end{aligned}$$

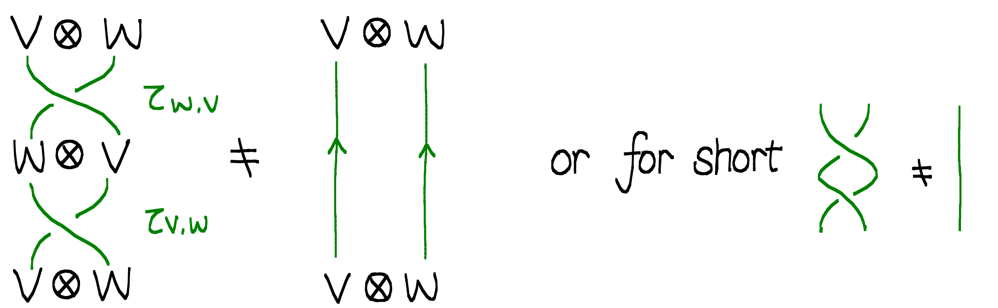
$\forall x \in \mathfrak{g}$, gives rise to a tensor product (monoidal) structure on $\mathrm{Mod}\, U(\mathfrak{g})$. This tensor product structure is symmetric in that the map

$$\begin{aligned} \tau_{V,W} : V \otimes W &\longrightarrow W \otimes V \\ v \otimes w &\longmapsto w \otimes v \end{aligned}$$

is a map of $U(\mathfrak{g})$-modules and doing it twice gives back identity:



Quantum group $U_q(\mathfrak{g})$ is a one parameter deformation of $U(\mathfrak{g})$ as a Hopf algebra, whose module category $\mathrm{Mod}\, U_q(\mathfrak{g})$ has the same family of simple objects as $\mathrm{Mod}\, U(\mathfrak{g})$. However, the comultiplication of $U_q(\mathfrak{g})$ is no longer cocommutative, so that the above $\tau_{V,W}$ won't be a map of $U_q(\mathfrak{g})$-modules any more. Instead, one has a non-trivial braiding on $U_q(\mathfrak{g})$-mod with the aid of R-matrices, still denoted τ:



Thus the module category of $U_q(\mathfrak{g})$ carries interesting braid group actions. The theory of quantum groups and their categorifications will be a main topic later.