§12. Categorification of the Heisenberg Algebra

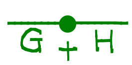
Biadjoint functors from finite groups

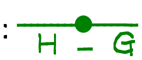
Let $\Bbbk$ be a field, G a finite group, and H a subgroup of G. We have inclusion of group algebras $\Bbbk[H] \subseteq \Bbbk[G]$, and thus adjoint functors:

$$\mathrm{Ind}_H^G \dashv \mathrm{Res}_G^H \dashv \mathrm{Coind}_H^G .$$

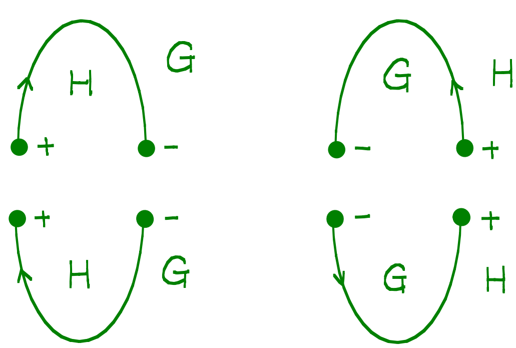
Moreover, under the finiteness assumption (this can be weakened to $[G:H] < \infty$), $\Bbbk[H] \subseteq \Bbbk[G]$ is a Frobenius extension, so that $\mathrm{Ind}_H^G \cong \mathrm{Coind}_H^G$, and Ind_H^G is biadjoint to Res_G^H. We will use the notation ${}_H G_G \triangleq {}_{\Bbbk[H]}\Bbbk[G]_{\Bbbk[G]}$, $G_H G = \Bbbk[G] \otimes_{\Bbbk[H]} \Bbbk[G]$ etc.

Biadjointness allows us to apply string notation introduced in §7. We will denote Ind_H^G / Res_G^H resp. by:

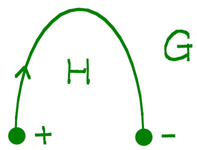
Ind_H^G : 

Res_G^H : 

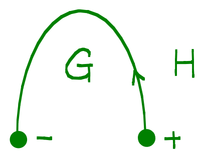
The biadjointness of them, are then given by oriented cups and caps:



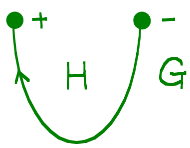
which are given explicitly by maps of bimodules:



$$G_H G \longrightarrow G, \qquad a \otimes b \longmapsto ab$$

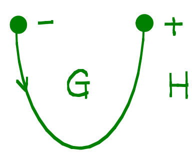


$${}_H G_H \xrightarrow{p_H} H, \qquad p_H(a) = \begin{cases} a & a \in H \\ 0 & a \notin H \end{cases}$$



$$G \longrightarrow G_H G, \qquad 1 \longmapsto \Sigma g_i^{-1} \otimes g_i$$

where $G = \sqcup_{i=1}^{m} H g_i$, and $\{g_i\}$ is a set of representatives in $H \backslash G$.



$$H \longrightarrow {}_H G_H, \qquad a \longmapsto a$$

The third map is well-defined since if $\{g_i'\}$ is another set of representative, then $g_i' = h_i g_i$ for some $h_i \in H$, so that

$$\Sigma_{i=1}^{n} g_i'^{-1} \otimes g_i' = \Sigma_i g_i^{-1} h_i^{-1} \otimes h_i g_i = \Sigma_i g_i^{-1} \otimes g_i .$$

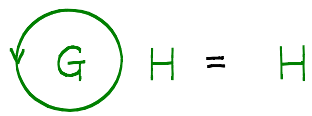
Furthermore, it is a (G,G)-bimodule map because $\forall g \in G$, $g_i g$ $h_{\varphi(i)} g_{\varphi(i)}$ where φ is a permutation of $\{g_1, \cdots, g_n\}$ and $h_{\varphi(i)} \in H$. Thus

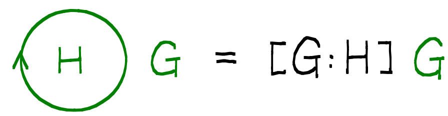
$$\begin{aligned}
(\Sigma_{i=1}^{n} g_i^{-1} \otimes g_i) g &= \Sigma_{i=1}^{n} g_i^{-1} \otimes h_{\varphi(i)} g_{\varphi(i)} \\
&= \Sigma_{i=1}^{n} g_i^{-1} h_{\varphi(i)} \otimes g_{\varphi(i)} \\
&= \Sigma_{i=1}^{n} g\, g_{\varphi(i)}^{-1} \otimes g_{\varphi(i)} \\
&= g\, (\Sigma_{i=1}^{n} g_i^{-1} \otimes g_i) .
\end{aligned}$$

The general theory in §7 gives us:

Thm. Ind_H^G, Res_G^H are acyclic biadjoint functors under these maps. □

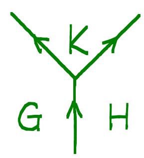
These maps gives us the relations:



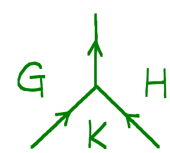


Now assume that $H \subseteq K \subseteq G$ are two subgroups. We depict by an orientend trivalent vertex the natural transformation:

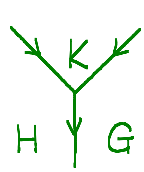
$\mathrm{Ind}_H^G \xrightarrow{\sim} \mathrm{Ind}_K^G \mathrm{Ind}_H^K$

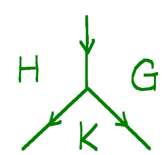


, $\mathrm{Ind}_K^G \mathrm{Ind}_H^K \xrightarrow{\sim} \mathrm{Ind}_H^G$



Similarly, for $\mathrm{Res}_K^H \mathrm{Res}_G^K = \mathrm{Res}_G^H$,





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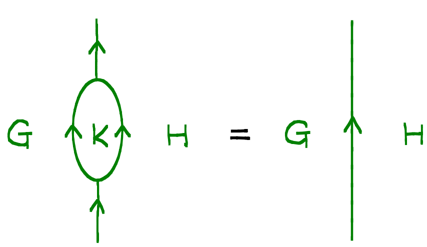
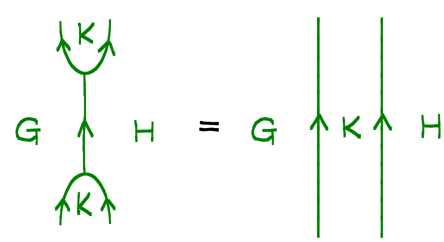
Then we have:



,



Next, we will give a graphical interpretation of Mackey's formula. Let $K \subseteq G \supseteq H$ be finite subgroups of G, we can decompose G into (K, H)-double cosets $G = \sqcup_{i \in I} K g_i H$, so that we have decomposition of $(\Bbbk[K], \Bbbk[H])$-bimodules:

$$_K\Bbbk[G]_H \cong \oplus_{i \in I}\ _K\Bbbk[K g_i H]_H .$$

Each $\Bbbk[K g_i H]$ has a simple description as a $(\Bbbk[K], \Bbbk[H])$-bimodule:

$$\begin{aligned} \Bbbk[K] \otimes \Bbbk[H] &\twoheadrightarrow \Bbbk[K g_i H] \\ k \otimes h &\longmapsto k g_i h \end{aligned}$$

and $k g_i h = k' g_i h' \Longleftrightarrow k^{-1} k' = g_i h h'^{-1} g_i^{-1} \in K \cap g_i H g_i^{-1} \triangleq L_i$, so that we have an isomorphism of bimodules:

$$\Bbbk[K] \otimes_{\Bbbk[L_i]} \Bbbk[H] \xrightarrow{\cong} \Bbbk[K g_i H]$$

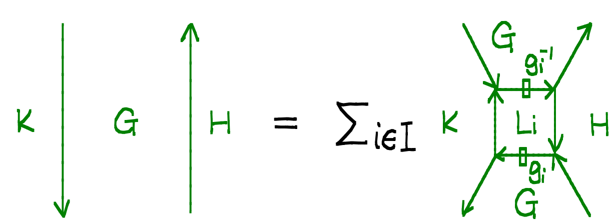
where L_i acts on K by right multiplication, and on H by a "g_i-twist":

$$\ell \cdot h = (g_i^{-1} \ell g_i) h$$

Then Mackey's formula follows:

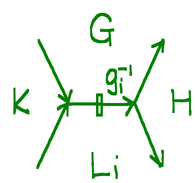
$$\begin{aligned} \mathrm{Res}^K_G \circ \mathrm{Ind}^G_H &= {}_K\Bbbk[G]_H \otimes_{\Bbbk[H]} - \\ &= \oplus_{i \in I}\ {}_K\Bbbk[K g_i H]_H \otimes_{\Bbbk[H]} - \\ &= \oplus_{i \in I}\ {}_K\Bbbk[K] \otimes_{\Bbbk[L_i]} \Bbbk[H] \otimes_{\Bbbk[H]} - \\ &= \oplus_{i \in I}\ \mathrm{Ind}^K_{L_i} \circ \mathrm{Res}^{L_i}_H . \end{aligned}$$

It acquires the following graphical interpretation:

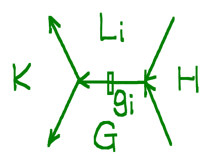


where the label g_i on an arrow indicates the isomorphism of $\mathrm{Ind}^G_{L_i}$ or $\mathrm{Res}^G_{L_i}$ coming from inclusions $L_i \hookrightarrow G$ and $g_i^{-1} L_i g_i \hookrightarrow G$.

Note that the diagrams below come from (K,H)-bimodule maps:



$$\begin{array}{ccc} \Bbbk[G] & & kg_ih \\ \uparrow & & \uparrow \\ \Bbbk[K]\otimes_{\Bbbk[L_i]}\Bbbk[H] & & k\otimes h \end{array}$$



$$\begin{array}{c} \Bbbk[Kg_iH] \cong \Bbbk[K]\otimes_{\Bbbk[L_i]}\Bbbk[H] \\ \uparrow \text{ projection} \\ \Bbbk[G] \end{array}$$

Problem: Investigate Mackey's theorem for systems of groups that naturally appear in geometry and number theory, such as $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ or $\pi_1(\text{Manifolds})$.

Symmetric groups

Now we will apply our general theory above to the special case of symmetric groups.

We will further simplify our notation $Res^{S_{n-1}}_{S_n}$, $Ind^{S_{n+1}}_{S_n}$ to R^{n-1}_n, I^{n+1}_n and (bi-)modules ${}_{S_n}(S_m)_{S_k}$ to ${}_n(m)_k$ etc.

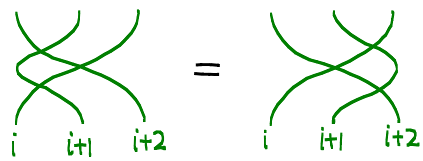
In §5, we have shown for Nil-Coxeter ring that

$$ {}_{NC_n}(NC_{n+1})_{NC_n} \cong NC_n \oplus (NC_n \otimes_{NC_{n-1}} NC_n)$$

which in turn gives us

$$\mathrm{Res}_{n+1} \circ \mathrm{Ind}_n \cong \mathrm{Id}_n \oplus \mathrm{Ind}_{n-1} \circ \mathrm{Res}_n$$

The proof there is just a version of Mackey's formula, and we only needed the "RIII" relation:



to obtain the decomposition of bimodules above. Thus for $\Bbbk[S_n]$, we also have:

$$_n(n+1)_n \cong {}_n(n)_n \oplus (n_{n-1}n)$$

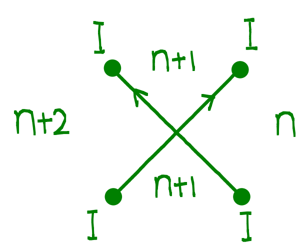
so that we also have:

$$R^n_{n+1} \circ I^{n+1}_n \cong \mathrm{Id}_n \oplus I^n_{n-1} \circ R^{n-1}_n .$$

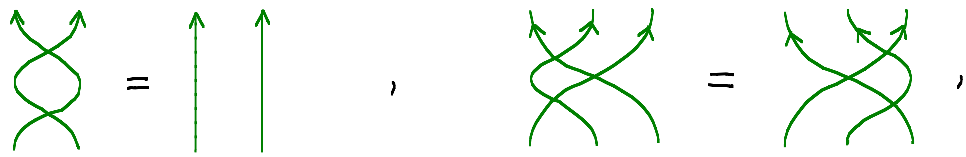
Next, notice that the functor $I^2 = I^{n+2}_{n+1} \circ I^{n+1}_n$ admits an endomorphism coming from the $(n+2, n)$-bimodule map

$$\begin{array}{ccc} _{n+2}(n+2)_n & & g s_{n+1} \\ \uparrow & & \uparrow \\ _{n+2}(n+2)_n & & g \end{array}$$

where $s_{n+1} = (n+1, n+2)$. We will denote this endomorphism by:

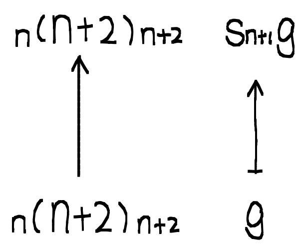


Then we have the relations:

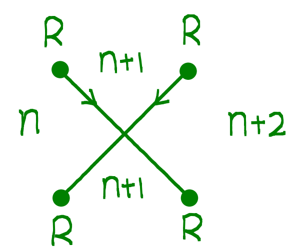


which follows from the corresponding relations in S_n.

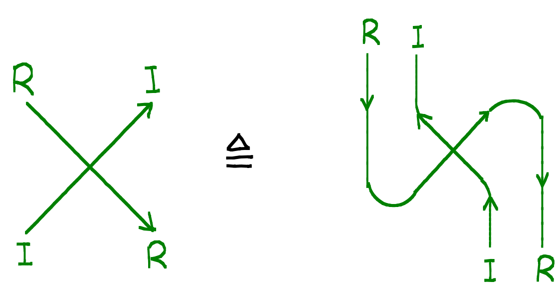
Similarly, $R^2 = R^{n+1}_{n+2} \circ R^{n}_{n+1}$ has as an endomorphism S_{n+1}:



which we depict as:

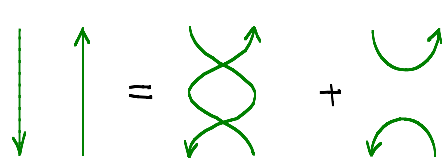


Using cups and caps we can produce crossings between R and I as well:

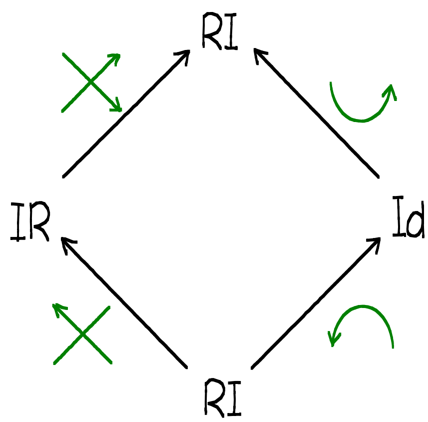


etc.

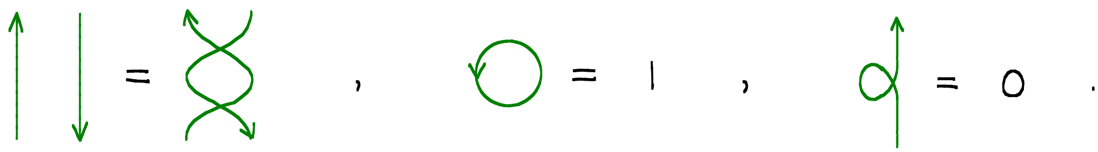
By the discussion above, we have,



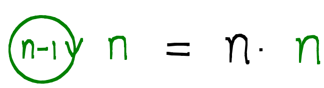
which is encoded in $RI \cong IR \oplus Id$ as follows:



and the relations (exercise):



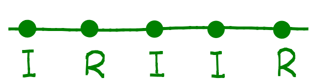
These are relations that do not depend on. Some relations, however, do depend on n. For instance,



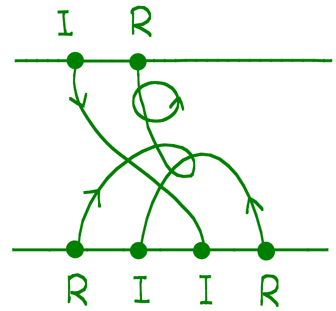
The monoidal category $\mathcal{H}$

Now we define an abstract monoidal $\Bbbk$-linear category $\mathcal{H}'$.

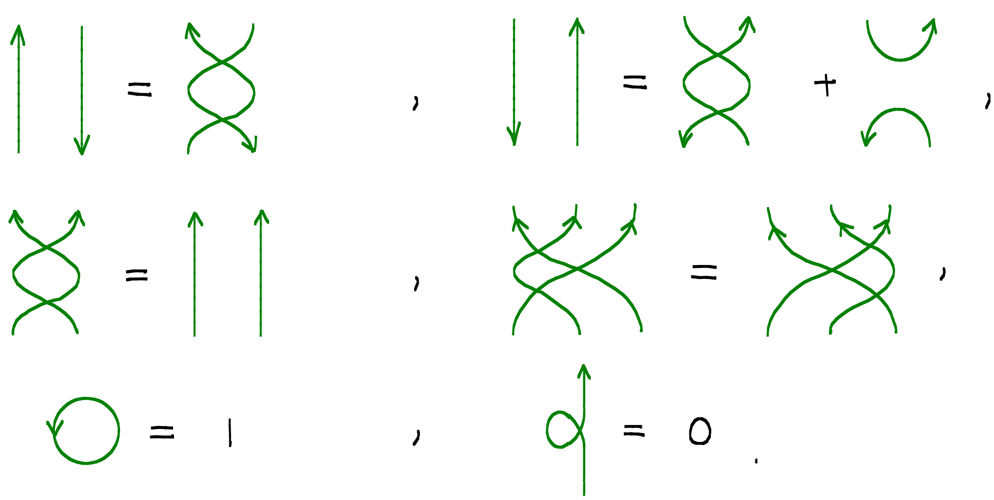
Objects of $\mathcal{H}'$ are defined to be finite direct sums of tensor products of I or R:



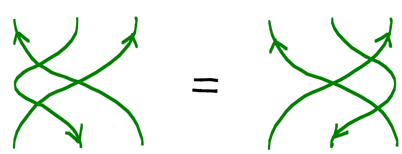
Morphisms of $\mathcal{H}'$ between objects are $\Bbbk$-linear combinations of oriented string diagrams, with at most simple crossings:



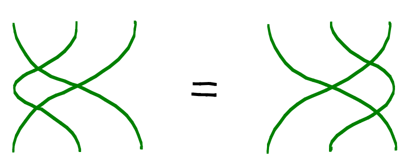
The morphisms are required to satisfy istopies relative to boundaries and local relations modeled on those relations above for symmetric groups which do not depend on n:



One can check that these relations imply:

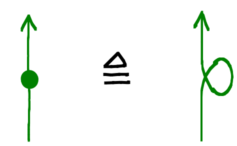


which further implies that

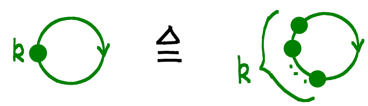


holds with arbitrary orientation.

The right curl doesn't simplify, and it will be convenient to relabel it as a dot:

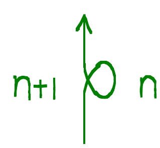


Moreover, clockwise circles carrying dots do not simplify:



The dot is related to the Jucy-Murphy elements in $\Bbbk[S_n]$ as follows.

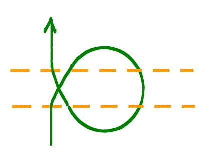
Lemma. Fix an $n \in \mathbb{N}$, we have:



$$= \sum_{i=1}^{n} (i, n+1) \triangleq J_n,$$

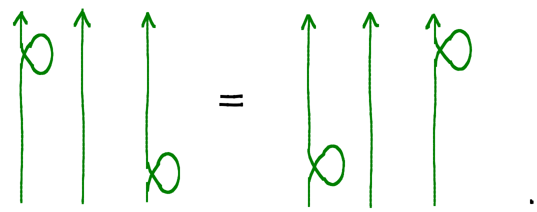
the n-th Jucy-Murphy elements.

Pf: An easy computation by decomposing the right curl into a cup, a crossing, and a cap:



□

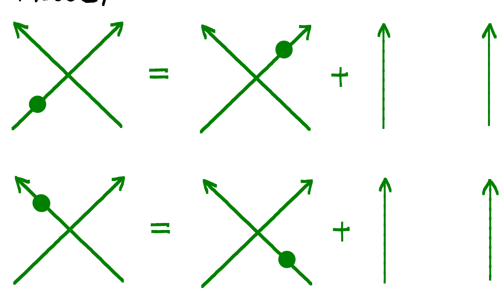
We recall that Jucy-Murphy elements commute with each other. For each $n \in \mathbb{N}$, $J_0 = 0$, $J_1 = (12)$, $\cdots$, J_{n-1} form a maximal commutative subalgebra of $\Bbbk[S_n]$ if $\operatorname{char} \Bbbk = 0$. The commutativity of these elements now becomes planar isotopy relations:



Rmk: Jucy-Murphy elements play very important roles in the representation theory of symmetric groups. See A. Okounkov, A. Vershik, A New Approach to the Representation Theory of the Symmetric Groups.

The following lemma is an easy consequence of the defining relations of $\mathcal{H}'$:

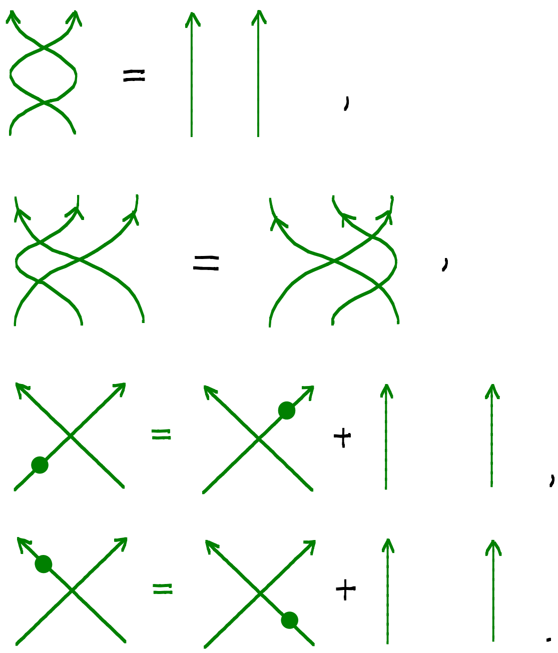
Lemma. In $\mathcal{H}'$, we have,



□

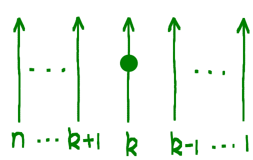
This relation, together with all the upward pointing relations, reminds us of the notion of the degenerate affine Hecke algebra:

Def. (Degenerate AHA on n-strands DH_n). DH_n is the $\Bbbk$-linear diagrammatic algebra on n strands (like NC_n) carrying dots subject to the local relations:



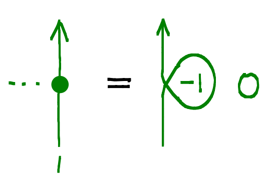
Notice that $\Bbbk[S_n]$ naturally embeds into DH_n as a subalgebra. Moreover, we have a retraction by assigning a dot on the k-th strand the k-th Jucy-Murphy element:

$$DH_n \longrightarrow \Bbbk[S_n]$$



$$\longmapsto J_k$$

Notice that

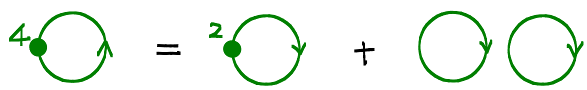


$$\longmapsto 0 \in \Bbbk[S_n].$$

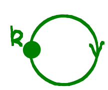
Now we will state the first main result about $\mathcal{H}'$. Before that, we state the following lemma, which followes from an induction argument.

Lemma. The conter-clockwise circle carrying dots can be reduced to polynomial combinations of clockwise circles with dots on them.

For instance,



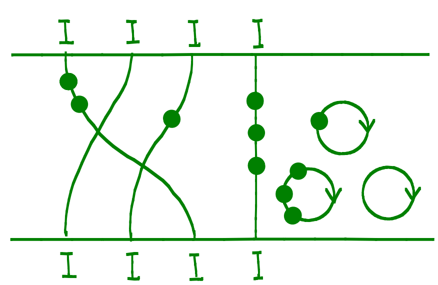
Thm. (1). $\mathrm{End}_{\mathcal{H}'}(\mathbb{1}_{\mathcal{H}'}) \cong \Bbbk[c_0, c_1, c_2, \cdots]$, where

$c_k =$ 

In other words, this is saying that any closed diagram in $\mathcal{H}'$ can be reduced to linear combinations of pictures with only clockwise circles with dots.

(2). $\mathrm{End}_{\mathcal{H}'}(I^n) \cong DH_n \otimes \mathrm{End}_{\mathcal{H}'}(\mathbb{1}_{\mathcal{H}'})$.

In other words, any pictures going from n bottom I's to n top I's can be reduced to linear combinations of elements of DH_n with some circles attached on their right hand side.



$n=4$

Notice that by turning everything above upside-down we get the analogous results for R^n.

For the proof, see M. Khovanov, Heisenberg algebra and a graphical calculus.

Def. The category $\mathcal{H}$ is defined to be the Karoubi envelope of $\mathcal{H}'$.

$\mathcal{H}$ is a $\Bbbk$-linear, monoidal category since $\mathcal{H}'$ is. By the thm above, $\Bbbk[S_n] \subseteq DH_n$ acts on I^n (R^n) $\in Ob(\mathcal{H}')$. Let e_n^+ (e_n^-) be the complete symmetrizer (anti-symmetrizer) in $\Bbbk[S_n]$ (char $\Bbbk = 0$), and define $A_n \triangleq (I^n, e_n^+)$, $B_n = (R^n, e_n^-) \in Ob(\mathcal{H}')$.

Prop. We have in $\mathcal{H}$ that

(1). $A_0 = B_0 = Id \quad A_n = B_n = 0$ if $n<0$.

(2). $RI \cong IR \oplus Id$

(3). $A_n A_m \cong A_m A_n$, $B_n B_m \cong B_m B_n$

(4). $B_m A_n \cong A_n B_m \oplus A_{n-1} B_{m-1}$

Hence $K_0(\mathcal{H})$ is a ring, in which

$$[B_m]\cdot[A_n] = [A_n][B_m] + [A_{n-1}][B_{m-1}]$$

These elements $[A_n]$, $[B_n]$ can be shown to generate the Heisenberg algebra

$$H = \Bbbk\langle p_n, q_n\rangle_{n\geq 0} / (p_n p_m = p_m p_n,\ q_n q_m = q_m q_n,\ p_n q_m = q_m p_n + \delta_{nm})$$

Thm. There is an injection

$$\gamma : H \longrightarrow K_0(\mathcal{H})$$

Conjecture: γ is an isomorphism.

The proofs can be found in the above mentioned paper.

Rmk: There is another categorification of H by Cautis-Licata, where they essentially used the rings $\Bbbk[S_n] \ltimes T^{\otimes n}$, where $T = \Lambda^2\mathbb{C}^2 \rtimes \Bbbk[G]$, and G is a finite subgroup of $SU(2)$. The algebra T describes the derived categories of coherent sheaves on some Nakajima quiver varieties (Kapranov).