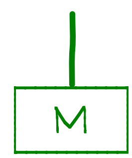
§10. Hochschild Homology and Applications to Link Homology I

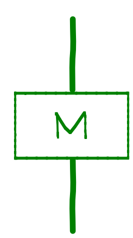
Hochschild (co-)homology

For simplicity, we fix a base field $\Bbbk$ and work with algebras over $\Bbbk$.

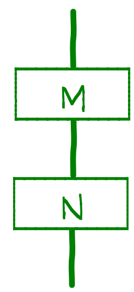
Let A be such an algebra, we introduce the following graphical depiction of any module M over A. M is depicted by a box with wires, and A operates on M via attaching elements of A onto M.



A bimodule ${}_AM_A$, which can be regarded as a module over $A \otimes_{\Bbbk} A^{op} \triangleq A^e$, will be depicted as labelled boxes with two wires:



where A operates on top and A^{op} operates from below.
In this graphical notation, the tensor product of bimodules over A will then be depicted as joining wires:



But $\otimes_A$ is in general not exact, so we will only use this picture to stand for $M \otimes^{\mathbb{L}}_A N$.

We now recall the definition of Hochschild (co-)homology, which are derived versions of (co-)invariants of a bimodule.

Def. (1). The invariants M^A of an A-bimodule M is the submodule

$$M^A \triangleq \{ m \in M \mid am = ma,\ \forall a \in A \} \cong \mathrm{Hom}_{A^e}(A, M).$$

(2). The coinvariants M_A of an A-bimodule M is the quotient module

$$M_A \triangleq M/[A, M] \cong M \otimes_{A^e} A,$$

where $[A, M]$ is the A^e-submodule generated by elements of the form $(am - ma)$.

(3). The Hochschild cohomology is the right derived functor of taking invariants:

$$HH^*(A,M) \triangleq R^*\mathrm{Hom}_{A^e}(A,M) = \mathrm{Ext}^*_{A^e}(A,M)$$

(4). The Hochschild homology is the left derived functor of taking coinvariants:

$$HH_*(A,M) \triangleq L^*(M \otimes_{A^e} A) = \mathrm{Tor}^{A^e}_*(A,M)$$

Rmk: It follows from this definition that

$$\begin{cases} HH^0(A,M) = M^A \\ HH_0(A,M) = M_A \end{cases}$$

Theoretically, we can compute Hochschild (co-) homology by resolving A by projective A^e-modules. This is done via the bar resolution:

$$\begin{tikzcd} \mathrm{Bar}(A): \cdots \arrow[r] & A^{\otimes n} \arrow[r, "d_n"] & \cdots \arrow[r] & A^{\otimes 3} \arrow[r] & A^{\otimes 2} \arrow[r] \arrow[d, "m"] & 0 \\ & & & 0 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

where the differential is given by:

$$d_n: A^{\otimes n} \longrightarrow A^{\otimes n-1}$$
$$a_1 \otimes \cdots \otimes a_n \mapsto \sum_{i=0}^{n-1} (-1)^i a_1 \otimes \cdots \otimes a_{i-1} a_i \otimes \cdots \otimes a_n$$

One checks readily that the augmentment complex

$$\mathrm{Bar}(A) \xrightarrow{m} A \longrightarrow 0$$

is contractible, where a homotopy h is given by

$$\begin{aligned} A^{\otimes n} &\xrightarrow{h} A^{\otimes n+1} \\ a_1 \otimes \cdots \otimes a_n &\longmapsto 1 \otimes a_1 \otimes \cdots \otimes a_n \ , \end{aligned}$$

so that $\mathrm{Bar}(A) \xrightarrow{m} A$ is a quasi-isomorphism of complexes of bimodules. Note that when $n \geq 2$,

$$A^{\otimes n} = A \otimes A^{\otimes n-2} \otimes A$$

is a free bimodule. Hence the bar resolution is a resolution of A by free bimodules.

When A is regular, we can find much simpler resolutions of A as bimodules:

E.g. $A \cong \Bbbk[x]$. We have the 2-step Koszul resolution:

$$\begin{aligned} 0 \longrightarrow \Bbbk[x] \otimes \Bbbk[x] &\longrightarrow \Bbbk[x] \otimes \Bbbk[x] \xrightarrow{m} \Bbbk[x] \longrightarrow 0 \\ 1 \otimes 1 &\longmapsto 1 \otimes x - x \otimes 1 \end{aligned}$$

An important special case is when $M = A$. By the remark at the end of the def., $HH^0(A) = Z(A)$ is just the center of

A. In general, $HH^*(A,A) \cong \mathrm{Ext}^*_{A^e}(A,A)$, equipped with the Yoneda pairing:

$$\mathrm{Ext}^i_{A^e}(A,A) \times \mathrm{Ext}^j_{A^e}(A,A) \longrightarrow \mathrm{Ext}^{i+j}_{A^e}(A,A)$$

becomes a super-commutative algebra. It acts on $D(A)$ as follows. For any $M \in D(A)$ and $\alpha \in HH^i(A,A)$, α is given by a map of chain complexes of A-bimodules:

$$\alpha : A \longrightarrow A[i]$$

in $D(A^e)$. Tensor this with M gives us

$$\alpha : M \longrightarrow M[i] \ \in \mathrm{Hom}_{D(A)}(M, M[i]) = \mathrm{Ext}^i(M,M)$$

The assignment is natural in M. In other words, α is a natural transformation of endo-functors of $D(A)$:

$$\alpha : \mathrm{Id}_{D(A)} \Longrightarrow \mathrm{Id}_{D(A)}[i].$$

Recall that in an abelian category $\mathcal{A}$, $Z(\mathcal{A}) \triangleq \mathrm{End}(\mathrm{Id}_{\mathcal{A}})$ is called the center of $\mathcal{A}$. Then for the derived category $D(\mathcal{A})$, we have the graded center

$$Z(D(\mathcal{A})) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}(\mathrm{Id}, \mathrm{Id}[i]),$$

and what we have exhibted is a map:

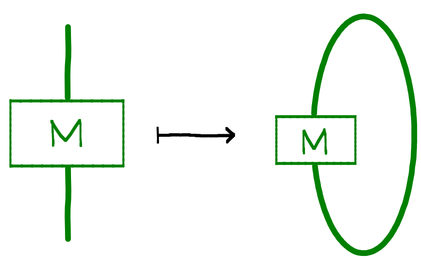
$$HH^*(A,A) \longrightarrow Z(D(A))$$

Problem: Is this map surjective / injective ?

Exercise: Show that $HH^1(A,A) = \mathrm{Der}(A,A) / \text{Inner derivations}$.

In what follows, we will depict taking the Hochschild homology of a (complex of) A-bimodule by closing off the diagram of M by joining the top and bottom wires:

$$M \longmapsto HH_*(A,M)$$



Since this is the exact functor that returns with a $\Bbbk$-vector space that the top A action can be transfered to the bottom A^{op}-action.

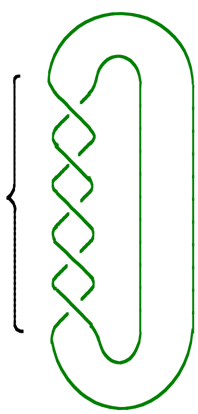
Relation to link homology

This starts with a simple observation of Przytycki. Consider the algebra $A = \Bbbk[x]/(x^2)$ which we used to define $\mathfrak{sl}_2$-link homology. As a bimodule over itself, it has an infinite free resolution:

$$\cdots \longrightarrow A\otimes A \xrightarrow[1\otimes 1 \,\mapsto\, x\otimes 1 - 1\otimes x]{} A\otimes A \xrightarrow{1\otimes 1 \,\mapsto\, x\otimes 1 + 1\otimes x} A\otimes A \xrightarrow[1\otimes 1 \,\mapsto\, x\otimes 1 - 1\otimes x]{} A\otimes A \xrightarrow{m} A \longrightarrow 0 \quad (*)$$

as a simplification of the bar resolution. In particular, the complex is 2-periodic. This phenomenon also occurs in the $\mathfrak{sl}_2$-theory. Consider the $(2,n)$ torus link as the closure of the braid:

n crossings

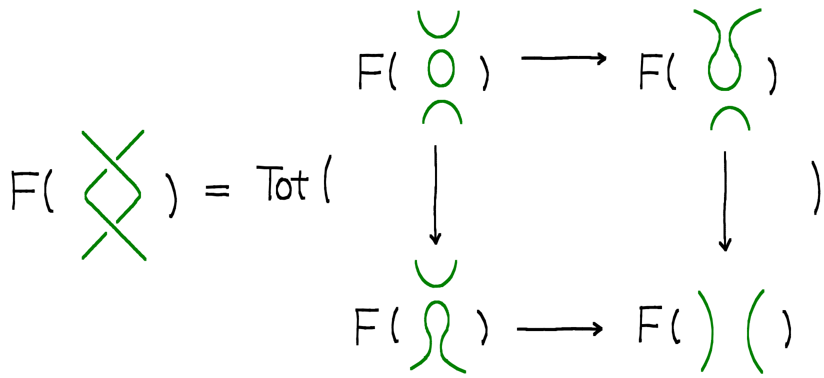


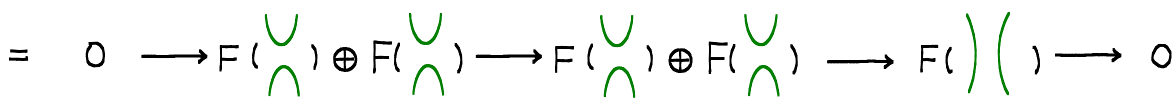
Recall that to a positive crossing, we assigned the complex of $H^1 = A$-bimodule maps (see §5.):

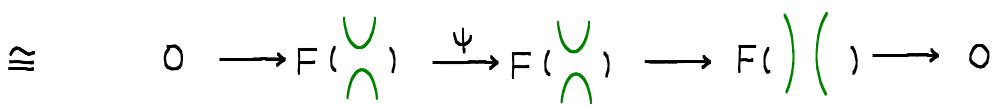
$$F(\times) \triangleq 0 \longrightarrow F(\asymp) \xrightarrow{m} F(\,)(\,) \longrightarrow 0,$$

where m is F applied to the saddle cobordism, and $F(\,)(\,)$ is the identity functor of H^1-modules, i.e. $H^1 \otimes_{H^1} -$, while $F(\asymp)$

is given by $H' \otimes_{\Bbbk} H' \otimes_{H'} -$. For two crossings:

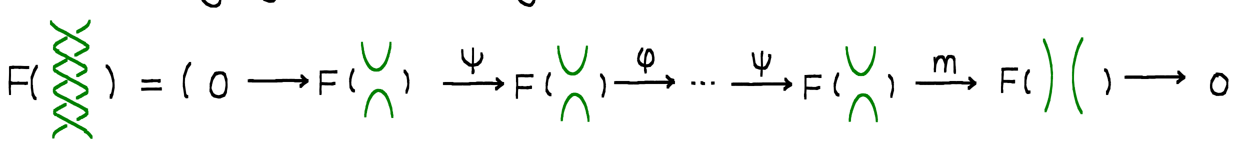






Here one checks that the map ψ is given by multiplication by the element $x \otimes 1 - 1 \otimes x$.

Inductively, for n crossings, we can show that



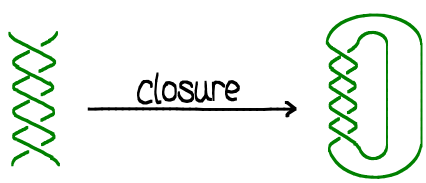
(**)

where φ is the multiplication by $x \otimes 1 + 1 \otimes x$, while ψ is the multiplication by $x \otimes 1 - 1 \otimes x$. Hence we have the following interesting:

Observation 1: $\boxed{\mathrm{Bar}(H') \cong \lim_{n\to\infty} F(\sigma^n)}$,

where $\sigma =$  .

Observation 2: The Hochschild homology of $H' = A$ can be identified as the limit of the $\mathfrak{sl}_2$-link homology of the $(2,n)$-torus link $\tilde{\sigma}^n$ as $n \longrightarrow \infty$.



σ^n $\tilde{\sigma}^n$

Indeed, to compute $HH_*(A,A)$, we use the bar resolution $(*)$, and tensor it with A. Notice that $x\otimes 1 + 1\otimes x$ becomes $2x$ while $x\otimes 1 - 1\otimes x$ becomes 0 in the resulting complex:

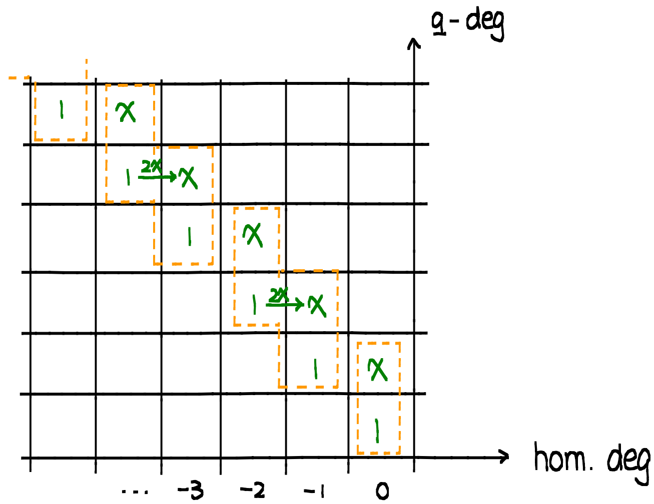
$$\left(\cdots \xrightarrow{x\otimes 1+1\otimes x} A\otimes A \xrightarrow{x\otimes 1-1\otimes x} A\otimes A \xrightarrow{x\otimes 1+1\otimes x} A\otimes A \xrightarrow{x\otimes 1-1\otimes x} A\otimes A \longrightarrow 0\right) \otimes_{A^e} A$$

$$= \cdots \xrightarrow{2x} A \xrightarrow{0} A \xrightarrow{2x} A \xrightarrow{0} A \longrightarrow 0$$

We see that the terms in homological degrees $[-n-1, 0]$ coincides with (**) :

$$0 \longrightarrow A \longrightarrow \cdots \xrightarrow{2x} A \xrightarrow{0} A \xrightarrow{2x} A \longrightarrow 0$$

Drawn on a grid diagram, the Hochschild complex looks like:

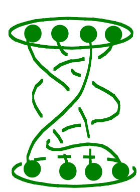


The graded Euler characteristic

$$\chi(HH_*(A,A)) = \chi(\lim_{n\to\infty} H(\tilde{\sigma}^n)) = 1$$

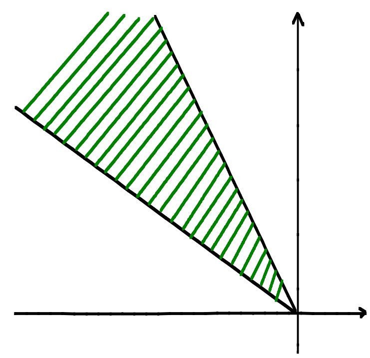
How far does the story extend

Now, let T be any tangle in $D \times I$ with the same even number of boundary points on $D \times \{0\}$ and $D \times \{1\}$.

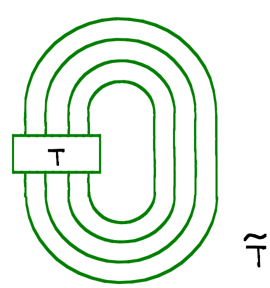


Previously, we have associated with it a complex of H^n-bi-modules, which is an invariant of tangles. So what can we say about $HH_*(H^n, F(T))$?

(1). It's always infinite dimensional, concentrated in a region like:

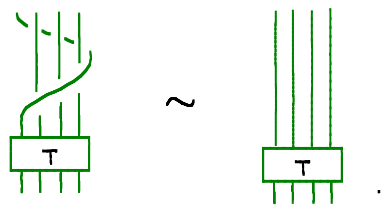


(2). It's a bigraded invariant of the tangle closure $\widetilde{T} \subseteq D \times S^1$



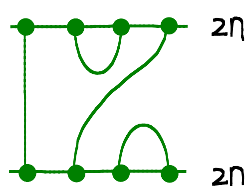
In fact, this is an invariant of T not only as embedded in $D \times S^1$, but also as $\widetilde{T} \subseteq S^2 \times S^1$. We will sketch a proof in what follows.

Recall that as tangles in $S^2 \times I$, there is one more move of tangles that results in isotopic tangle closures in $S^2 \times S^1$:

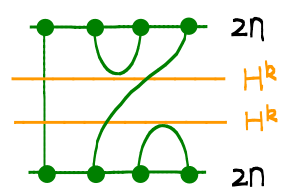


Thus we need to check the invariance of $HH_*(H^n, F(T))$ under this move.

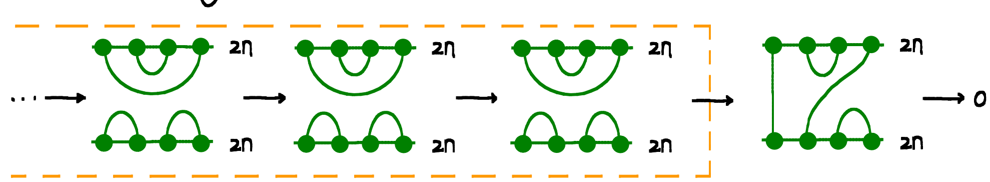
Now recall that $F(T)$ is built by first resolving T into flat tangles as



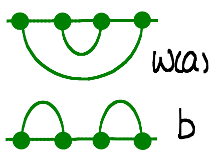
For each of such flat tangles, through its "thinest" part, we resolve it similarly as we did for | | as H^k-bimodules:



which is now infinite (bar-resolution):



This results in a complex of H^n projective bimodules of $P_a \otimes_{\Bbbk} {}_bP$ where $w(a)/b$ is the top/bottom matching

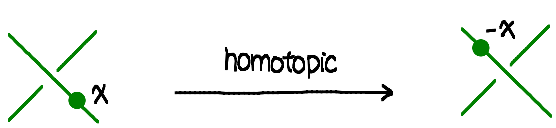


$$\longmapsto \qquad P_a \otimes_{\Bbbk} {}_bP$$

(see the notation in §5). We then collect all the infinite chains of bimodules into a total complex, which can be regarded as the bar resolution of $F(T)$ as an H^n-bimodule.

Then the invariance follows from checking for matchings, the move above introduces isotopies:

On differentials, it introduces an over-all sign since one can check that when x passes through a crossing, it changes sign:



,

so that the total chain complex remain homotopic. The invariance follows from these observations.

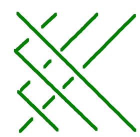
We summarize some properties about $HH_*(T) \triangleq HH_*(H^n, F(T))$, in analogy with the previous subsection:

(1). If $T = |||\cdots|$ on $2n$ strands, $F(T) = {}_{H^n}H^n{}_{H^n}$, and

$$\chi(HH_*(T)) = \frac{1}{n+1}\binom{2n}{n}$$

(2). $HH_*(T)$ satisfies the skein relations.

(3). $HH_*(T)$ is the "limit" of $\mathfrak{sl}_2$-homology of $\widetilde{T\cdot\sigma^n}$, where σ is the full braid twist on $2n$ strands:

$\sigma =$ 

and $\sim$ denotes the braid closure.

$$HH_*(T) = \lim_{n\to\infty} H(\widetilde{T\cdot\sigma^n})$$

Problem: Is $HH_*(T)$ functorial in $T \subseteq S^2 \times I$?