Introduction to algebraic topology, Spring 2013

Homework 4, due Tuesday, February 19

1. Which of the following are categories? In each example, composition of morphisms is the obvious one.

(a) Objects are finite sets, morphisms are injective maps of sets.
(b) Objects are sets, morphisms are surjective maps of sets.
(c) Objects are abelian groups, morphisms are isomorphisms of groups.
(d) Objects are sets, morphisms are maps of sets which are not surjective.
(e) Objects are topological spaces, morphisms are homeomorphisms.

2. (a) An object I of a category $\mathcal{C}$ is called *initial* if for any object X of $\mathcal{C}$ there exists a unique morphism from I to X. Which of the following categories have initial objects: category of sets, category of groups, category of topological spaces, category of infinite-dimensional vector spaces over a given field (morphisms are linear maps)? Show that any two initial objects in a category are isomorphic.

(b) By analogy, give the definition of a *terminal* object in a category. Which of the categories above admit terminal objects? How do initial and terminal objects compare in a category of sets?

3. Give a example of a category with

(a) one object and four morphisms;
(b) two objects and five morphisms.

4. Determine fundamental groups of the following spaces:

$$\mathbb{RP}^2 \times \mathbb{RP}^1, \quad \mathbb{RP}^\infty \times S^5, \quad \mathbb{RP}^2 \vee \mathbb{RP}^3.$$

5. Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 \setminus X)$.

6. Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

7. Give an example of a two-dimensional CW-complex whose fundamental group is (a) trivial, (b) $\mathbb{Z}/4$, (c) $\mathbb{Z}/2 \times \mathbb{Z}/4$, (d) $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

Extra credit:
Hatcher Section 2.1 exercise 10 (page 53).