Introduction to algebraic topology. Final exam, due Friday May 10, 2013.

Name: ____________________

☐ I certify that my answers and solutions are my own work.

Email me if you have any questions. You can solve problems in any order.

1. (10 points) Mark those squares that are followed by correct statements.

☐ Any compact topological space is homeomorphic to a finite CW-complex.

☐ If X is a CW-complex of dimension n then $H_{n+1}(X) = 0$.

☐ Punctured $\mathbb{RP}^3$ is homotopy equivalent to $\mathbb{RP}^2$.

☐ The Euler characteristic of $\mathbb{RP}^3$ is 1.

☐ There exists a space X such that $H_n(X) \neq 0$ for all $n \geq 0$.

2. (10 points) Group spaces listed below into equivalence classes with respect to the homotopy equivalence relation. For each homotopy equivalence class determine homotopy groups π_1, π_2 and homology groups H_n.
(a) $S^1 \vee S^1$,
(c) $S^1 \times [0,1] \times [0,1]$,
(b) The complement of three points in the two-sphere,
(d) The two-skeleton of a three-dimensional simplex,
(e) Wedge (one-point union) of a Möbius band and an annulus,

(f) Wedge of an annulus and the infinite-dimensional sphere S^∞,
(g) Twice-punctured universal cover of $\mathbb{RP}^2$,
(h) Universal cover of $\mathbb{R}^3 \setminus \{0\}$.

3. (10 points) For each of the two statements below either prove it or give a counterexample.

(a) A map f from X to Y induces an isomorphism on their zeroth homology groups if and only if f induces a bijection between the sets of path-connected components of X and Y.

(b) If the first homology group $H_1(X)$ of a connected CW-complex X is trivial (equal 0) then X is contractible.

4. (10 points) (a) Describe the fundamental group of the Klein bottle via generators and relations.

(b) Prove that the Klein bottle cannot be a retract of any space homotopy equivalent to the two-torus.

5. (10 points) Show that there does not exist a fiber bundle with the total space $S^1 \times S^2$, fiber S^1, and base – the two-torus.

6. (10 points) Describe how to construct a CW-decomposition of the suspension SX given a CW-decomposition of X. If X has n cells, how many cells will your decomposition have? Find a formula relating the Euler characteristics of X and SX, assuming X is a finite CW-complex.

7. (10 points) A CW-complex X has a cell decomposition with two 0-cells, one 1-cell, and one 3-cell. Is X necessarily path-connected? What can you say about homology groups of X and the fundamental group $\pi_1(X)$? Try to classify such X up to homotopy equivalence.

8. (10 points) Let X be the two-sphere S^2 with attached equatorial disk. Describe a cell decomposition of X. What are the homology groups of X? Consider the map $f : X \longrightarrow X$ that fixes every point on the equatorial disk and projects points in the two hemispheres onto the corresponding points of the disk along the z-axis. (If a point had coordinates (x, y, z), $x^2+y^2+z^2 = 1$, then $f(x, y, z) = (x, y, 0)$). Compute the induced map f_* on homology groups of X and determine the Lefshetz number of f. If a map g is homotopic to f, does g necessarily have a fixed point?

Extra credit: Compute the homology groups of the subspace of $I \times I$ consisting of the four boundary edges plus all points in the interior whose first coordinate is rational.