CHAPTER III

SPECTRAL SEQUENCES

§18. FILTRATION IN A SPACE AND ITS SPECTRAL SEQUENCE

Let X be a topological space with a sequence of subspaces X_i:

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_k = X.$$

If for example, X is a CW complex, one can take for $X_i = X^i$, the i-skeleton of X. Such sequence is called a (finite) *filtration* in X.

(Usually we add to this sequence the terms $X_{-2} = X_{-3} = \ldots = \emptyset$, and $X_{k+1} = X_{k+2} = \ldots = X$.)

In the following we shall investigate homology groups of filtered spaces. As a rule, the same arguments can be repeated for cohomology groups with minor changes.

Let $C_q(X)$ be the singular chains of X. Then obviously

$$0 \subset C_q(X_0) \subset C_q(X_1) \subset \ldots \subset C_q(X_{k-1}) \subset C_q(X_k) = C_q(X).$$

We shall say that $\alpha \in C_q(X)$ has filtration i if $\alpha \in C_q(X_i)$ and $\alpha \notin C_q(X_{i-1})$. Thus $C_q(X_i)$ contains all elements of filtration at most i and no others. In short, we shall say that on $C_q(X)$ a filtration is given.

Let us take

$$E_0^{i,q-i} = C_q(X_i, X_{i-1}) = C_q(X_i)/C_q(X_{i-1}).$$

The numbers i and q will be called *filtering degree* and *full degree*, respectively. The boundary operator ∂_q: $C_q(X_i, X_{i-1}) \to C_{q-1}(X_i, X_{i-1})$ introduced for relative chains will be denoted by $d_0^{i,q-1}$, i. e. $d_0^{i,q-1}$: $E_0^{i,q-1} \to E_0^{i,q-i-1}$. Obviously $d_0^{i,q-i} \circ d_0^{i,q-i+1} = 0$.

This way an algebraic complex has been obtained. Let us consider the homology groups, i. e. the groups $H_q(X_i, X_{i-1})$ and denote them by $E_1^{i,q-i}$.

Spectral sequences are defined so that each term is, in a certain sense, smaller than the preceding one, namely, is a homology group of it.

Definition (The subgroup $Z_r^{i,q-i} \subset E_0^{i,q-i}$).

Let $\alpha \in E_0^{i,q-i} = C_q(X_i)/C_q(X_{i-1})$. We shall say that $\alpha \in Z_r^{i,q-i}$ whenever the coset α contains some representative $a \in C_q(X_i)$ whose boundary has a filtration r units smaller than a, i. e. $\partial a \in C_{q-1}(X_{i-r})$.

Case $r=0$: $Z_0^{i,q-i} = E_0^{i,q-i}$.

Case $r=1$: there exists $a \in \alpha$ such that $\partial a \in C_{q-1}(X_{i-1})$ i. e. α is a cycle in

$C_q(X_i, X_{i-1})$, i. e. $Z_1^{i,q-i} = Z_q(X_i, X_{i-1})$.

Let us remark that $\alpha \in Z_q(X_i, X_{i-1})$ implies the property $\partial a \in C_{q-1}(X_{i-1})$ for every representative $a \in \alpha$. The same is not true, however, when $r > 1$, as it can easily be shown.

With r increasing the group is, obviously, decreasing and for sufficiently large r it reduces to $Z_q(X_i)/Z_q(X_{i-1})$. (This stable group is denoted by $Z_\infty^{i,q-i}$.) We obtain a chain of inclusions: $Z_\infty^{i,q-i} \subset \ldots \subset Z_{r+1}^{i,q-i} \subset Z_r^{i,q-i} \subset \ldots \subset Z_0^{i,q-i} = E_0^{i,q-i}$.

Definition (The subgroup $B_r^{i,q-i} \subset E_0^{i,q-i}$).

Let us consider an element $\alpha \in E_0^{i,q-i}$. We shall say that $\alpha \in B_r^{i,q-i}$ if and only if the coset α contains a representative $a \in C_q(X_i)$ such that $a = \partial b$, where $b \in C_{q+1}(X_{i+r-1})$.

What does it mean that $\alpha \in B_0^{i,q-i}$? It means that the coset $\alpha \in E_0^{i,q-i} = C_q(X_i, X_{i-1})$ contains some $a \in C_q(X_i)$ such that $a = \partial b$ where $b \in C_{q+1}(X_{i-1})$, i. e. $a \in C_q(X_{i-1})$, i. e. $\alpha = 0$. And so, $B_0^{i,q-i} = 0$.

What is $B_1^{i,q-i}$? If $\alpha \in B_1^{i,q-i}$, then the coset $\alpha \in E_0^{i,q-i} = C_q(X_i, X_{i-1})$ contains some representative $a \in C_q(X_i)$ such that $a = \partial b$, where $b \in C_{q+1}(X_i)$, i. e. $b_1^{i,q-i}$ is the subgroup of relative boundaries in $C_q(X_i, X_{i-1})$: $B_1^{i,q-i} \subset B_{r+1}^{i,q-i}$.

Obvious inclusion: $B_r^{i,q-i} \subset B_{r+1}^{i,q-i}$.

If r is increasing the group $B_r^{i,q-i}$ increases and for sufficiently large r it is equal to

$$B_\infty^{i,q-i} = B_q(X) \cap C_q(X_i) / B_q(X) \cap C_q(X_{i-1}).$$

Now we have the chain of inclusions:

$$0 = B_0^{i,q-i} \subset B_1^{i,q-i} \subset \ldots \subset B_r^{i,q-i} \subset B_{r+1}^{i,q-i} \subset \ldots \subset B_\infty^{i,q-i}.$$

The inclusion $B_\infty^{i,q-i} \subset Z_\infty^{i,q-i}$ is obvious.

Thus we have a chain of inclusions

$$\begin{array}{l} 0 = B_0^{i,q-i} \subset B_1^{i,q-i} \subset \ldots \subset B_r^{i,q-i} \subset B_{r+1}^{i,q-i} \subset \ldots \subset B_\infty^{i,q-i} \subset \\ \qquad\qquad \| \\ \qquad B_q(X_i, X_{i-1}) \\ \subset Z_\infty^{i,q} \subset \ldots \subset Z_{r+1}^{i,q-i} \subset Z_r^{i,q-i} \subset \ldots \subset Z_1^{i,q-i} \subset Z_0^{i,q-i} = \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ = E_0^{i,q-i} = C_q(X_i, X_{i-1}). \qquad\qquad Z_q(X_i, X_{i-1}) \end{array} \qquad (*)$$

Let us consider the quotient group $E_r^{i,q-i} = Z_r^{i,q-i}/B_r^{i,q-i}$ $(r = 0, 1, \ldots, \infty)$.

For $r = 0$ we have $E_0^{i,q-i} = Z_0^{i,q-i}/B_0^{i,q-i} = E_0^{i,q-i}/\{0\} = C_q(X_i, X_{i-1})$. And so, $E_0^{i,q-i} = C_q(X_i, X_{i-1})$ i. e. $E_0^{i,q-i}$ is the very group defined above and denoted by the same symbol. Further,

$$E_1^{i,q-i} = Z_1^{i,q-i}/B_1^{i,q-i} = Z_q(X_i, X_{i-1})/B_q(X_i, X_{i-1}) = H_q(X_i, X_{i-1}).$$

In the chain $(*)$ the groups decrease as r increases: the denominator is increasing while the numerator is decreasing. Obviously there exists a number r_0 such that $E_{r_0}^{i,q-i} = E_{r_0+1}^{i,q-i} = \ldots = E_\infty^{i,q-i}$ for all i and q.

Definition (The differential $d_r^{i,q-i}: E_r^{i,q-i} \to E_r^{i-r,q+r-i-1}$).

Let $\alpha \in E_r^{i,q-i} = Z_r^{i,q-i}/B_r^{i,q-i}$ and let $\alpha' \in Z_r^{i,q-i}$ be a representative of α. Assume that $a \in C_q(X_i)$ represents $\alpha' \in Z_r^{i,q-i} \subset C_q(X_i)/C_q(X_{i-1})$ and $b = \partial a$ has filtration at most $i-r$. Then the coset β' of b in the group $C_{q-1}(X_{i-r})/C_{q-1}(X_{i-r-1})$ belongs to $Z_r^{i-r,q+r-i-1}$ and defines in $E_r^{i-r,q+r-i-1}$ an element depending only on α. Let this element be denoted by $d_r^{i,q-i}\,\alpha$.

We leave it to the reader to check the correctness of this definition, show that $d_r^{i,q-i}$ is a homomorphism and prove the equality $d_r^{i-r,q+r-i-1} \circ d_r^{i,q-i} = 0$.

The homomorphism $d_1^{i,q-i}: E_1^{i,q-i} \to E_1^{i+1,q-i}$ coincides with $\partial: H_q(X_i, X_{i-1}) \to$ $\to H_{q-1}(X_{i-1}, X_{i-2})$ in the exact sequence of the triple (X_i, X_{i-1}, X_{i-2}). (The definitions of these homomorphisms are, word for word, the same.)

Let us define E_r by taking $E_r = \bigoplus_{i,q} E_r^{i,q-i}$. Then the differentials $d_r^{i,q-i}$ yield a differential $d_r: E_r \to E_r$, $d_r \circ d_r = 0$.

The sequence of the groups E_r and the differentials d_r is called a *spectral sequence.*

Theorem. E_{r+1} is the homology group of E_r with respect to the differential d_r. That is, $E_{r+1} = \operatorname{Ker} d_r/\operatorname{Im} d_r$. Moreover, $E_{r+1}^{i,q-i} = \operatorname{Ker} d_r^{i,q-i}/\operatorname{Im} d_r^{i+r,q-i-r+1}$.

Proof. (We keep the notations of the definition.)

Assume that $d_r^{i,q-i}(\alpha) = 0$. Then β' belongs to $B_r^{i-r,q-i+r-1}$ i. e. there exists a representative $c \in \beta'$ such that $c = \partial\tau$, where $\tau \in C_q(X_{i-1})$.

The element a is a representative of α' and $a \in C_q(X_i)$. Consider $a-\tau \in C_q(X_i)$. We remind that $C_q(X_{i-1}) \subset C_q(X_i)$. By subtracting τ we leave the coset in $C_q(X_i)/C_q(X_{i-1})$ unaltered, i. e. $a-\tau$ is a representative of the same coset $\alpha' \in Z_r^{i,q-i}$. As the differential $d_r^{i,q-i}$ is correctly defined (it does not depend on the choice of the representative), therefore $a-\tau$ could have been chosen from the beginning, instead of a. Thus $\partial(a-\tau(\in C_{q-1}(X_{i-r-1})$ and $\alpha' \in Z_{r+1}^{i,q-i}$. Let the coset of α' in $E_{r+1}^{i,q-i}$ be noted by $\bar{\alpha}$. By assigning $\bar{\alpha}$ to α we obtain a homomorphism $\operatorname{Ker} d_r^{i,q-i} \to E_{r+1}^{i,q-i}$. Now it remained to prove that

(1) the homomorphism is correctly defined, that is $\bar{\alpha}$ depends on α alone;

(2) it is an epimorphism;

(3) the kernel is the group $\operatorname{Im} d_r^{i-r,q-i+r-1}$.

This part of the proof will be left to the reader.

Let us now consider the "stabilizing" group $E_\infty^{i,q-i}$. By definition $E_\infty^{i,q-i} =$ $= Z_\infty^{i,q-i}/B_\infty^{i,q-i}$.

Let us denote by ${}_{(i)}H_q(X)$ the image of the homomorphism $H_q(X_i) \to H_q(X)$ induced by the inclusion $X_i \subset X$. We obtain a filtration

$$0 = {}_{(-1)}H_q(X) \subset {}_{(0)}H_q(X) \subset \ldots \subset {}_{(k)}H_q(X) = H_q(X).$$

Theorem. $E_\infty^{i,q-i} = {}_{(i)}H_q(X)/{}_{(i-1)}H_q(X)$.

Proof. By definition

$$Z_\infty^{i,q-i} = Z_q(X_i)/Z_q(X_{i-1}),$$

$$B_\infty^{i,q-i} = B_q(X) \cap C_q(X_i)/B_q(X) \cap C_q(X_{i-1}),$$

$$_{(i)}H_q(X) = Z_q(X_i)/B_q(X) \cap C_q(X_i),$$

$$_{(i-1)}H_q(X) = Z_q(X_{i-1})/B_q(X) \cap C_q(X_{i-1}).$$

The required equality

$$Z_\infty^{i,q-i}/B_\infty^{i,q-i} = {}_{(i)}H_q(X)/{}_{(i-1)}H_q(X)$$

immediately follows from the following obvious algebraic statement: If A and B are subgroups of a group then $(A+B)/B = A/(A \cap B)$.

Let us now summarize the results.

Theorem. If the space is filtered by the subspaces $X_i: \emptyset = X_{-1} \subset X_0 \subset \ldots \subset X_{k-1} \subset \subset X_k = X$, then there exist groups $E_r^{p,q}$ for every non-negative r and every p and q (where $E_r^{p,q} = 0$ for $p<0$ and $p>k$), and homomorphisms $d_r^{p,q}: E_r^{p,q} \to E_r^{p-r,q+r-1}$, $d_r^{p-r,q+r-1} \circ d_r^{p,q} = 0$, such that

(1) $E_r^{p,q} = \operatorname{Ker} d_r^{p,q}/\operatorname{Im} d^{p+r,q-r+1}$,

(2) $E_0^{p,q} = C_{p+q}(X_p, X_{p-1})$,

(3) $E_\infty^{p,q} = \dfrac{\operatorname{Im}(H_{p+q}(X_p) \to H_{p+q}(X))}{\operatorname{Im}(H_{p+q}(X_{p-1}) \to H_{p+q}(X))} = \dfrac{_{(p)}H_{p+q}(X)}{_{(p-1)}H_{p+q}(X)}$.

This statement is the *Leray's theorem*.

Let us explain the statement (3). It says that for every m the group $H_m(X)$ contains a subgroup $_{(0)}H_m(X) = E_\infty^{0,m}$; the quotient group $H_m(X)/E_\infty^{0,m}$ contains a subgroup $E_\infty^{1,m-1}$ and so on, and, at last, the quotient group

$$(\ldots((H_m(X)/E_\infty^{0,m})/E_\infty^{1,m-1})/E_\infty^{2,m-2}\ldots)/E_\infty^{k-1,m-k+1}$$

is equal to $E_\infty^{k,m-k}$. The group $\oplus_{p+q=m} E_\infty^{p,q}$ is, therefore, closely related to $H_m(X)$; it is said to be adjoint to $H_m(X)$ relatively to the filtration $_{(i)}H_m(X)$ and is sometimes denoted by $GH_m(X)$.

Let us note some formal properties of adjoint groups. Let A be an Abelian group, $0 \subset A_0 \subset A_1 \subset \ldots \subset A_m = A$ a filtration and $GA = \oplus_i A_i$ where $A_i^0 = A_i/A_{i-1}$ is the adjoint group.

(1) If GA is finitely generated, then so is A and their ranks are equal.

(2) If GA is finite, then so is A and their orders are equal.

(3) If all but one A_i^0 are zero groups, then $GA = A$.

(4) If all but two A_i^0 ($A_{i_1}^0$ and $A_{i_2}^0$, where $i_1 > i_2$) are zero, then $A_{i_1}^0 \subset A$ and $A/A_{i_1}^0 = A_{i_2}^0$.

(5) If all A_i^0 are free Abelian groups, then GA is isomorphic to A.

(6) If all A_i^0 are vector spaces over some field k, then GA is isomorphic to A.

The proof is left to the reader.

By cohomology substituted for homology, a similar theory can be built up. The final result is the following.

Theorem. If the space is filtered by the subspaces X_i (i.e. $\emptyset = X_{-1} \subset X_0 \subset \ldots \subset \subset X_{k-1} \subset X_k = X$), then there exist groups $E_r^{p,s}$ defined for $r \geq 0$ and for every p and q (where $E_r^{p,q} = 0$ for $p < 0$ and $p > k$) and homomorphisms $d_r^{p,q}: E_r^{p,q} \to E_r^{p+r,q-r+1}$ (where $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$) such that

(1) $E_r^{p,q} = \operatorname{Ker} d_r^{p,q} / \operatorname{Im} d_r^{p-r,q+r-1}$,

(2) $E_0^{p,q} = C^{p+q}(X_p, X_{p-1})$,

(3) $$E_\infty^{p,q} = \frac{\operatorname{Ker}(H^{p+q}(X) \to H^{p+q}(X_{p-1}))}{\operatorname{Ker}(H^{p+q}(X) \to H^{p+q}(X_p))} = \frac{{}_{(p-1)}H^{p+q}(X)}{{}_{(p)}H^{p+q}(X)},$$

i. e. $\oplus_{p+q=m} E_\infty^{p,q}$ is adjoint with $H^m(X)$ with respect to the filtration

$$0 = {}_{(k)}H^m(X) \subset \ldots \subset {}_{(0)}H^m(X) \subset {}_{(-1)}H^m(X) = H^m(X),$$

where ${}_{(i)}H^m(X)$ stands for the kernel of the mapping $H^m(X) \to H^m(X_i)$ induced by the inclusion $X_i \subset X$.

The proof, as we said, is similar to that of the homology theorem; one has to introduce the filtration $0 = {}^{(k)}C^q(X) \subset \ldots \subset {}^{(0)}C^q(X) \subset {}^{(-1)}C^q(X) = C^q(X)$ to the group $C^q(X)$ such that ${}^{(i)}C^q(X)$ consists of the cochains γ: $C_q(X) \to \mathbf{Z}$ such that $\gamma(c) = 0$ whenever $c \in C_q(X_i) \subset C_q(X)$. Furthermore, $E_0^{i,q-i} = C^q(X_i, X_{i-1}) = {}^{(i-1)}C^q(X) / {}^{(i)}C^q(X)$ is taken and the boundary operator ∂ will be substituted by the coboundary operator δ wherever it occurs.

Finally, the reader can prove the analogous statements both for homologies and cohomologies in the more general case when coefficients are taken in an arbitrary group.

The first example: a new understanding of computation of the homology groups of CW complexes

Let X be a CW complex and let it be filtrated by its skeletons, $X_k = X^k$. Then $E_0^{p,q} = \mathscr{C}_{p+q}(X^p, X^{p-1})$ and

$$E_1^{p,q} = H_{p+q}(X^p, X^{p-1}) = \begin{cases} 0 & \text{for } q \neq 0, \\ \mathscr{C}_p(X) & \text{for } q = 0; \end{cases}$$

$$d_1^{p,q}: E_1^{p,q} \to E_1^{p-1,q}$$

i. e. $d_1^{p,q}$ is a homomorphism of zero groups if $q \neq 0$ and of $\mathscr{C}_p(X)$ into $\mathscr{C}_{p-1}(X)$ if $q = 0$.

In virtue of the remark following the definition of the differentials, the last homomorphism coincides with the homomorphism of the exact sequence of the triple (X^p, X^{p-1}, X^{p-2}) i. e. with the boundary homomorphism ∂: $\mathscr{C}_p(X) \to \mathscr{C}_{p-1}(X)$. We obtain $E_2^{p,q} = 0$ if $q \neq 0$ and $E_2^{p,0} = H_p(X)$.

Furthermore $d_2^{p,q}: E_2^{p,q} \to E_2^{p-2,q+1}$. Now either $E_2^{p,q}$ or $E_2^{p-2,q+1}$ is equal to zero, and thus $d_2^{p,q} = 0$ and $E_3^{p,q} = E_2^{p,q}$. In the same way we get $E_3 = E_4 = \ldots = E_\infty$.

No line $p+q=n$ contains more than one group different from zero: $E_\infty^{n,0}=E_2^{n,0}$. Hence, by property (3) of the adjointness, $E_\infty^{n,0}=H_n(X)$.

Thus the spectral sequence of a CW complex filtrated by its skeletons (considered over any Abelian group of coefficients) is trivial.

(Remark. A spectral sequence will be called *trivial* if every differential d_r is equal to zero for $r\geq 2$, i. e. $E_2=E_3=E_4=\ldots=E_\infty$.)

The second example: a new understanding of the homology sequence of a pair

Let us consider a two-termed filtration: $\emptyset\subset Y\subset X$ where $X_0=Y$ and $X_1=X$; $p=0,1$. Now $E_0^{p,q}=C_{p+q}(X_p, X_{p-1})$ and

$$E_1^{p,q}=H_{p+q}(X_p, X_{p-1})=\begin{cases} H_p(Y) & \text{if } p=0,\\ H_{p+1}(X, Y) & \text{if } p=1.\end{cases}$$

Of the groups $E_1^{p,q}$ only $E_1^{0,q}$ and $E_1^{1,q}$ are different from zero. As for the differentials the only one *a priori* different from zero is $d_1^{1,q}\colon E_1^{1,q}\to E_1^{0,q}$; for $p\neq 1$ the differential $d_1^{p,q}$ is trivial by consideration of the dimensions. (From now on "consideration of the dimensions" will mean "by taking into account the indices p, q and r of the group $E_r^{p,q}$ and of the differential $d_r^{p,q}$.")

The homomorphism $d_1^{1,q}\colon E_1^{1,q}\to E_1^{0,q}$ coincides with $\partial\colon H_{q+1}(X, Y)\to H_q(Y)$ in the exact sequence of the pair (X, Y). We have $E_2^{1,q}=\operatorname{Ker}\partial$, $E_2^{0,q}=H_q(Y)/\operatorname{Im}\partial$. For the rest, $E_2^{p,q}=0$. Hence, by consideration of the dimensions, all differentials d_r, $r\geq 2$ are trivial, i. e. $E_2^{p,q}=E_\infty^{p,q}$. Thus $E_2^{1,q}=E_\infty^{1,q}$ and $E_2^{0,q}=E_\infty^{0,q}$.

Now E_∞ is known to be related to $H_*(X)$ and to the filtration $0\subset\operatorname{Im}H_q(Y)\subset H_q(X)$ in the following sense.

$$E_\infty^{0,q}=\operatorname{Im}H_q(X_0)/\operatorname{Im}H_q(X_{-1})=\operatorname{Im}H_q(Y),$$

$$E_\infty^{1,q}=\operatorname{Im}H_{q+1}(X_1)/\operatorname{Im}H_{q+1}(X_0)=H_{q+1}(X)/\operatorname{Im}H_{q+1}(Y)$$

or

$$\operatorname{Ker}\partial=H_{q+1}(X)/\operatorname{Im}H_{q+1}(Y),\qquad H_q(Y)/\operatorname{Im}\partial=\operatorname{Im}H_q(Y).$$

Let us now consider the exact homology sequence of the pair (X, Y):

$$\ldots\to H_q(X, Y)\xrightarrow{\partial}H_{q-1}(Y)\to H_{q-1}(X)\to H_{q-1}(X, Y)\to\ldots$$

Obviously this sequence is equivalent to the two equalities above. We conclude that the spectral sequence of a two-termed filtration is equivalent to the exact sequence of the pair.

Exercise. Compute the spectral sequence of a three-termed filtration and prove that it is equivalent to the exact sequence of the triple.

This far we have only considered finite filtrations. The constructions can be carried out for infinite filtrations $\emptyset = X_1 \subset X_0 \subset X_1 \subset \ldots \subset X_\infty = X$, too. The first difficulty arises at the definition of the group $E_\infty^{p,q}$. In fact the groups $E_N^{p,q}$ do not necessarily stabilize as N is growing. In the case of cohomology, for $r > p$, the group $E_{r+1}^{p,q}$ is isomorphic to the kernel of the differential $d_r^{p,q}$ ($\operatorname{Im} d_r^{p-r,q+r-1} = 0$ follows from dimensional considerations). Hence $E_{p+1}^{p,q} \supset E_{p+2}^{p,q} \supset \ldots$ and we can set $E_\infty^{p,q} = \bigcap_{i>0} E_{p+i}^{p,q}$. In the homological case, for $r > 0$, $E_{r+1}^{p,q}$ is similarly isomorphic to the quotient group $E_r^{p,q}/\operatorname{Im} d_r^{p+r,q-r+1}$ and $E_\infty^{p,q}$ is defined as the limit of the sequence $E_{p+1}^{p,q} \to E_{p+2}^{p,q} \to \ldots$.

Statement (3) of the Leray theorem is valid in this case without any modification. The filtration ${}_{(i)}H_m(X)$, however, will be infinite and a special proof is required to show the equality $\bigcup_{i=0}^{\infty} {}_{(i)}H_m(X) = H_m(X)$, which is valid if the filtration $X_0 \subset X_1 \subset \ldots \subset X_\infty = X$ satisfies the following additional condition: *for every compact set $K \subset X$ there exists a finite index k such that $K \subset X_k$*. (The skeleta filtration of a CW complex satisfies this condition automatically.)

Under the same condition the filtration ${}_{(i)}H^m(X)$ defined in the cohomology groups has the property $\bigcap_{i=0}^{\infty} {}_{(i)}H^m(X) = 0$.

The proof is left to the reader.

§19. THE SPECTRAL SEQUENCE OF A FIBRATION

Let p: $E \to B$ be a Serre fibration with B being a connected CW complex. (The assumption that B is a connected CW complex is actually unnecessary, as the construction, following below, can be carried out to any topological space. Indeed, for any topological space X there exists a CW complex X' and a mapping f: $X' \to X$ such that f induces isomorphisms between the homotopy groups. Thus any given base B can be substituted by a CW complex B' and an f: $B' \to B$. The fibration p: $E \to B$ is likewise substituted by p': $E' \to B'$ induced from it by f.)

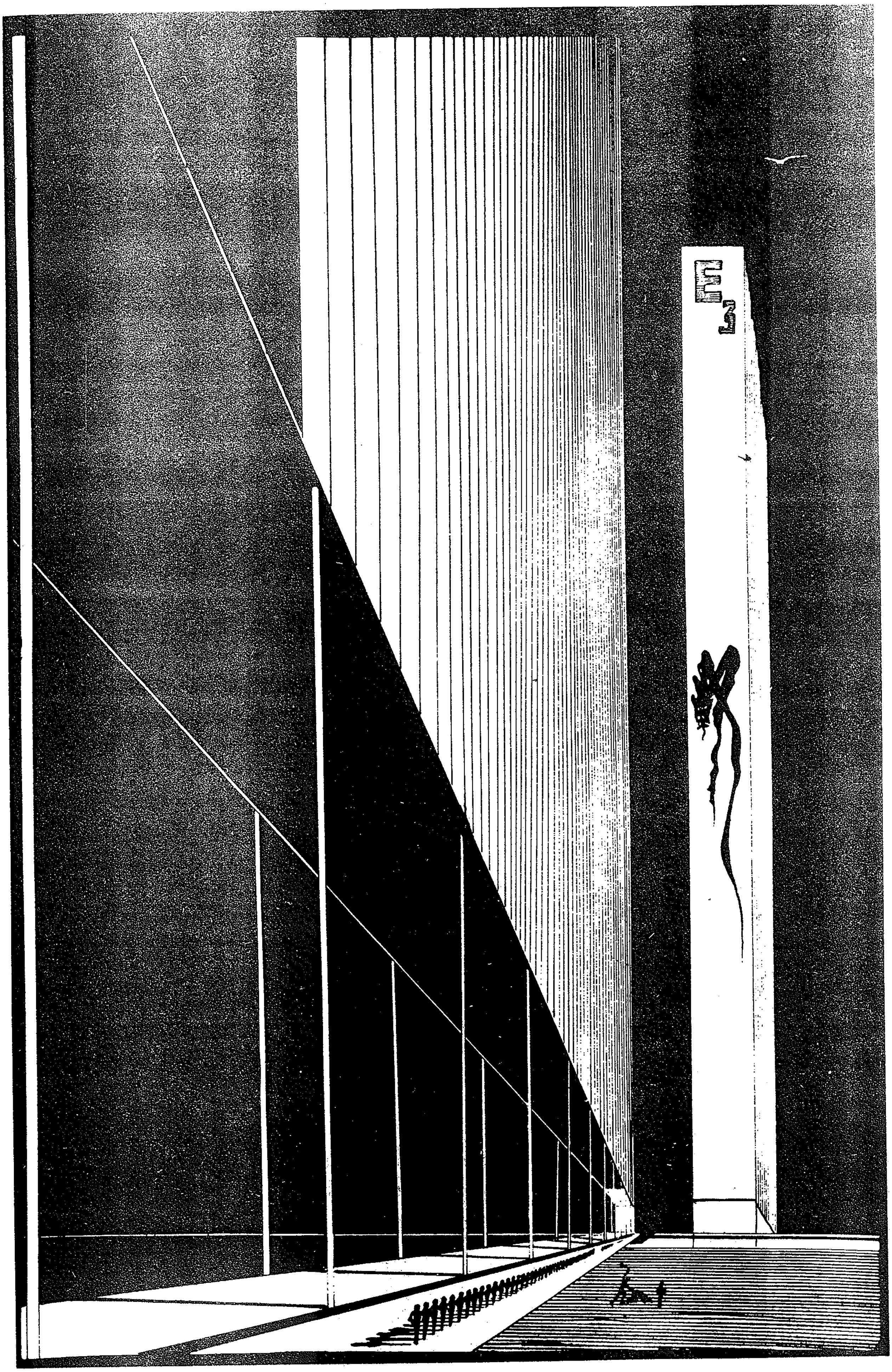
Our next aim is to find the homology and cohomology groups of the space E, assuming that those of B and F are known.

Let B^α denote the α-skeleton of B. Let B be filtrated by its skeletons: $\emptyset \subset B^{-1} \subset B^0 \subset \subset B^1 \subset \ldots \subset B^{n-1} \subset B^n = B$. The projection mapping induces a filtration of E:

$$\emptyset = E^{-1} \subset E^0 \subset E^1 \subset \ldots \subset E^{n-1} \subset E^n = E$$

where $E^i = p^{-1}(B^i)$. We shall consider the spectral sequence of E generated by this filtration.

As it will turn out the term E_2 of the sequence can be expressed in terms of the homology groups of the base and the fibre, i. e. the spectral sequence is rather strictly, though not completely, determined by properties of B and F.



By definition, $E_0^{p,q} = C_{p+q}(E^p, E^{p-1})$. (We are going to consider the case of homologies. Cohomologies can be treated in quite the same way.)

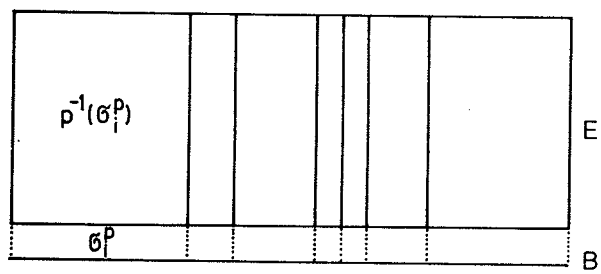
Again, $E_1^{p,q} = H_{p+q}(E^p, E^{p-1}) = H_{p+q}(E^p/E^{p-1})$. We shall show that $H_{p+q}(E^p/E^{p-1}) \approx \mathscr{C}_p(B; H_q(F))$ (we consider the cellular chains of the CW complex B).

We begin with describing E^p/E^{p-1}. The difference $E^p \setminus E^{p-1}$ consists of the pre-images $p^{-1}(\sigma_i^p)$ where σ_i^p are the p-dimensional cells of B. These sets are open in E^p and pairwise disjoint. Therefore

$$E^p/E^{p-1} = \vee_i E^p/(E^p \setminus p^{-1}(\sigma_i^p)) = \vee_i \overline{p^{-1}(\sigma_i^p)}/p^{-1}(\dot{\sigma}_i^p) =$$
$$= \vee_i p^{-1}(\bar{\sigma}_i^p)/p^{-1}(\dot{\sigma}_i^p)$$

where

$$\dot{\sigma}_i^p = \bar{\sigma}_i^p \setminus \sigma_i^p.$$



For the sake of simplicity the fibration will be assumed to be locally trivial. It will be left to the reader to prove the statement for the general case of Serre fibrations.

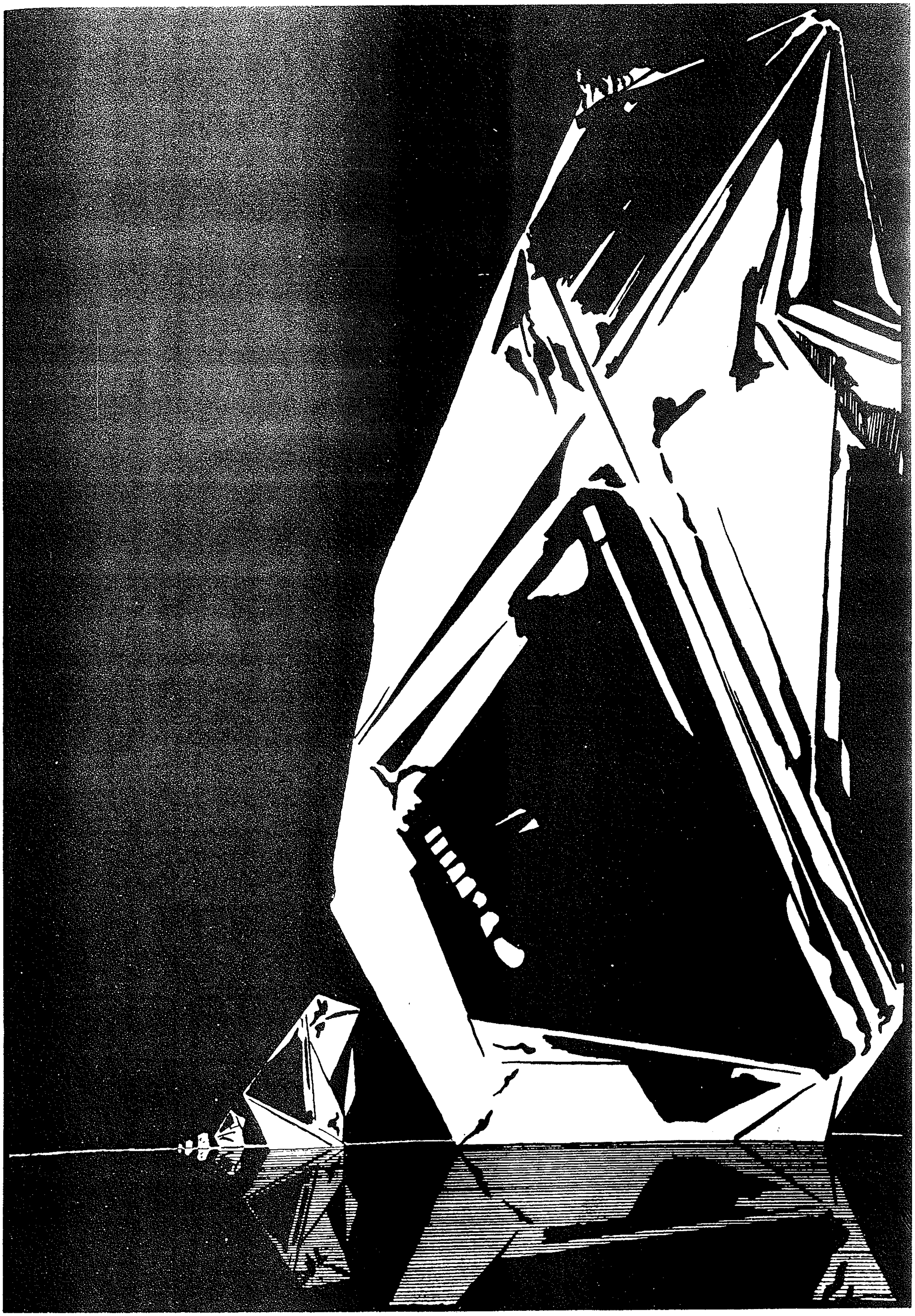
Let us show that $p^{-1}(\bar{\sigma}_i^p)/p^{-1}(\dot{\sigma}_i^p) \approx D^p \times F/S^{p-1} \times F$.

Let f: $D^p \to B$ be the characteristic mapping of the cell σ_i^p. (We use here the notation D^p for the p-ball, because the symbol B^p denotes here the p-skeleton of the base B.) The fibration $\bar{E} \to D^p$ induced from p: $E \to B$ by f is trivial, as any fibration over D^p, i. e. $\bar{E} \approx D^p \times F$.

$$\begin{array}{ccccccc} \bar{E}^{\cdot} & \subset & \bar{E} & \to & p^{-1}(\bar{\sigma}_i^p) & \subset & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow p \\ S^{p-1} & \subset & D^p & \to & \bar{\sigma}_i^p & \subset & B \end{array}$$

Since the image of f is $\bar{\sigma}_i^p$, this mapping can be decomposed to $D^p \to \bar{\sigma}_i^p \subset B$ which results a decomposition $\bar{E} \to p^{-1}(\bar{\sigma}_i^p) \subset E$ of the corresponding mapping $\bar{E} \to E$. The mapping $\bar{E} \to p^{-1}(\bar{\sigma}_i^p)$ maps $\bar{E} \setminus \bar{E}^{\cdot}$ homeomorphically onti $p^{-1}(\sigma_i^p)$ and maps $\bar{E}^{\cdot}$ onto $p^{-1}(\dot{\sigma}_i^p)$ (the latter restriction not being a homeomorphism). Hence

$$p^{-1}(\bar{\sigma}_i^p)/p^{-1}(\dot{\sigma}_i^p) = \bar{E}/\bar{E}^{\cdot} = (D^p \times F)/(S^{p-1} \times F).$$



Let us now turn to the proof of the statement.

$$\mathscr{C}_p(B;\ H_q(F)) = \text{(by definition) } H_p(B^p,\ B^{p-1};\ H_q(F)) =$$
$$= \text{(by definition)} \oplus H_p(\bar{\sigma}_i^p,\ \dot{\sigma}_i^p;\ H_q(F)) =$$
$$= \oplus H_p(\bar{\sigma}_i^p/\dot{\sigma}_i^p;\ H_q(F)) = \oplus H_q(F),$$
$$H_{p+q}(E^p/E^{p-1}) = H_{p+q}(\vee_i p^{-1}(\bar{\sigma}_i^p)/p^{-1}(\dot{\sigma}_i^p)) =$$
$$= \oplus H_{p+q}(p^{-1}(\sigma^p)/p^{-1}(\dot{\sigma}^p)) =$$
$$= \oplus H_{p+q}(D^p \times F/S^{p-1} \times F),$$

where the direct sums are taken over all i-cells.

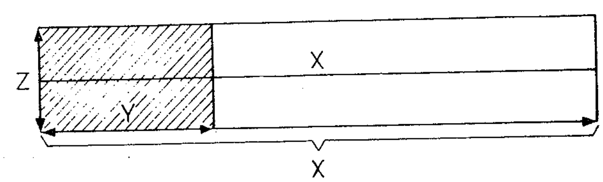
Let us show that $H_{p+q}(D^p \times F/S^{p-1} \times F) = H_q(F)$. It will follow then that $H_{p+q}(E^p/E^{p-1}) = \oplus H_q(F)$ i. e. $H_{p+q}(E^p/E^{p-1}) = \mathscr{C}_p(B;\ H_q(F))$. As it can easily be seen the cohomology groups of X and ΣX coincide up to a shift by one of the dimensions, i. e., for $q>0$, $H_q(X) = H_{q+1}(\Sigma X)$, therefore $H_{q+p}(\Sigma^p X) = H_q(X)$.

Were $D^p \times F/S^{p-1} \times F$ the (p-th) suspension over F, we should have the required statement. It is not the case, however. Actually the suspension $\Sigma^p f$ can be obtained from the space in consideration by additional factorization:

$$\Sigma^p F = (D^p \times F/S^{p-1} \times F)/S^p.$$

(Here $S^p \subset D^p \times F/S^{p-1} \times F$ is the sphere obtained from $D^p \subset D^p \times F$ by pasting together the points of $S^{p-1} \subset \mathscr{D}^p$.)

Let us prove it. Let X, Y and Z be arbitrary spaces, $Y \subset X$. As it can easily be seen on the picture,



$$X \times Z/(Y \times Z) \cup X = (X/Y) \times Z/Z \vee (X/Y).$$

In our case $X = D^p$, $Y = S^{p-1}$, $Z = F$ i. e.

$$(D^p \times F/S^{p-1} \times F)/S^p = S^p \times F/F \vee S^p. \quad \text{“smash” } S^p \wedge F$$

Let us consider the space $S^p \times F/S^p \vee F$.

This space is called the tensor product of S^p and F and is denoted by $S^p \otimes F$ (one defines the tensor product of two arbitrary spaces X and Y in the same way).

Let us prove $S^p \otimes F = \Sigma^p F$. First, the tensor product is associative: $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ (the verification of this is left to the reader), second, as it can easily be seen, $S^1 \otimes F = \Sigma F$ and $S^p = \underbrace{S^1 \otimes \ldots \otimes S^1}_{p}$, i. e. $S^p \otimes F = (\underbrace{S^1 \otimes \ldots \otimes S^1}_{p}) \otimes F =$

$$= (\underbrace{S^1 \otimes \ldots \otimes S^1}_{p-1}) \otimes \Sigma F = \ldots = \Sigma^p F.$$

Thus

$$((D^p \times F)/(S^{p-1} \times F))/S^p = \Sigma^p F.$$

For $q>0$ it is obvious that $H_{p+q}(D^p \times F/S^{p-1} \times F)$ = (from the exact sequence of the pair) $H_{p+q}(D^p \times F/S^{p-1} \times F,\ S^p) = H_{p+q}(\Sigma^p F) = H_q(F)$ and that has been the statement. The case $q=0$ is left to the reader. As well as $q=1$!

If F is a CW complex the equality $H_{p+q}(D^p \times F, S^{p-1} \times F) = H_q(F)$ can be verified in the following simpler way. The complex $D^p \times F$ has three kinds of cells according to the construction of D^p as a three-cell complex: two of them belong to $S^{p-1} \times F$ and they are therefore ignored as cohomology is considered. The remaining cells are in a one-to-one correspondence with the cells belonging to F, only their dimensions are p units larger.

The fact we just have proved enables us to determine the first term of the spectral sequence of the space E filtrated by the subspaces E^p:

$$E_1^{p,q} = \mathscr{C}_p(B;\ H_q(F)).$$

It is worthwhile to pay a little more attention to this equality. Let $\Sigma \alpha_i \sigma_i^p$, where $\alpha_i \in H_q(F)$, be an element of the group $\mathscr{C}_p(B;\ H_q(F))$. The element of $E_1^{p,q} = \oplus_i H_{p+q}(p^{-1}(\bar{\sigma}_i^p)/p^{-1}(\dot{\sigma}_i^p))$ corresponding to it is constructed by using same homeomorphisms of the standard object $D^p \times F/S^{p-1} \times F$ onto $p^{-1}(\bar{\sigma}_i^p)/p^{-1}(\dot{\sigma}_i^p)$, fixed for each i. This homeomorphism is not unique, even in terms of homotopy, and it is important to know the one that has been applied. It obviously suffices to fix the homeomorphism of the subspace $F \subset D^p \times F/S^{p-1} \times F$ which lies over the centre of the ball D^p, to the fibre $F_{x_0} \subset p^{-1}(\bar{\sigma}_i^p)/p^{-1}(\dot{\sigma}_i^p)$ over the centre x_0 of the cell σ_i^p. Once these homeomorphisms have been fixed, the isomorphism $E_1^{p,q} \approx \mathscr{C}_p(B;\ H_q(F))$ is determined.

Is it possible to choose the homeomorphism $F \approx F_{x_0}$ universally in some sense for every cell (up to homotopy)?

Each path connecting two points x_1 and x_2 of the base induces (up to homotopy) a homeomorphism $F_{x_1} \approx F_{x_2}$ while homotopic paths induce the same homeomorphism. If B is simply connected the homotopy class of this homeomorphism is totally independent of the particular path. Thus we fix $F \to F_{x_0}$ for some point x_0 and define the homeomorphisms $F \approx F_x$ for all $x \in B$ canonically (again up to homotopy). This procedure gives a well-defined isomorphism $E_1^{p,q} \approx \mathscr{C}_p(B;\ H_q(F))$. The same can be achieved if the base is allowed not to be simply connected but the fibration is *simple* (i. e. given any pair $x_1,\ x_2 \in B$ all paths connecting these points induce homotopic homeomorphisms $F_{x_1} \approx F_{x_2}$).

We will only study simple fibrations. In the general case we restrict ourselves to the basic formulations.

Let us now consider a fibration with simply-connected base (or a simple fibration). We have

$$d_1^{p,q}: E_1^{p,q} = \mathscr{C}_p(B;\ H_q(F)) \to E_1^{p-1,q} = \mathscr{C}_{p-1}(B;\ H_q(F)).$$

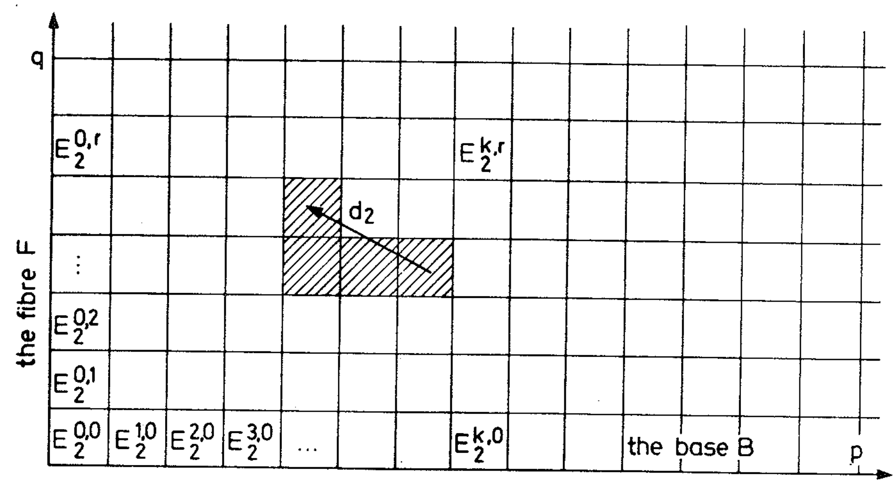
The reader can easily verify that this homomorphism is identical with the boundary homomorphism

$$\partial: \mathscr{C}_p(B; H_q(F)) \to \mathscr{C}_{p-1}(B; H_q(F)),$$

thus $E_2^{p,q} = H_p(B; H_q(F))$.

Remark. As it follows from the universal coefficient formula if the domain of the coefficients is a field, then $E_2^{p,q} = H_p(B) \otimes H_q(F)$. The same holds for integral homology under the condition that either $H_*(B)$ or $H_*(F)$ is torsion-free.

There exists a diagram very convenient for the illustration of the term $E_2 = \oplus_{p,q} E_2^{p,q}$ which will often prove helpful:



The arrow shows the action of the differential $d_2^{m,n}$ (the knight's progress). As r grows the arrows showing the action of the differentials $d_r^{m,n}$ grow, trying to coincide with the direction of the line $p+q = \text{const}$.

The botton row contains the groups $E_2^{k,0} = H_k(B; H_0(F))$, i. e. (if the fibre is connected) the homology groups of the base. The first column from the left contains the homologies of the fibre (provided that the base is connected). In the diagram for the E_∞ term, the line $p+q = m$ consists of groups whose sum is associated with $H_m(F)$.

The case of the cohomology spectral sequence of a filtration $\emptyset = E^{-1} \subset E^0 \subset \subset \ldots \subset E^{n-1} \subset E^n = E$ can be treated in the same way. If the fibration is simple, we have $E_2^{p,q} = H^p(B; H^q(F))$. A similar diagram describes the spectral sequence with the only difference that the arrows are directed in the opposite side.

Remark. The numeration of the groups belonging to the spectral sequence as given in §18 might have appeared a bit strange there. Now it seems justified.

§20. FIRST APPLICATIONS

The bare fact that a Serre fibration possesses a spectral sequence contains enough information to enable us to determine some homology groups. The number of these cases is not very large, nevertheless they illustrate the potential of this method quite convincingly.

Homology groups of the special unitary group $SU(n)$

The elements of this group are the transformations of the n-dimensional complex space, satisfying the well-known conditions. The group $SU(n-1)$ will be considered as a subgroup standardly imbedded in $SU(n)$. The homogeneous space $SU(n)/SU(n-1)$ is then nothing else than the sphere of real dimension $2n-1$, i. e. we have a fibration $SU(n) \xrightarrow{SU(n-1)} S^{2n-1}$ $(n \geq 2)$.

In the case $n=2$ the fibre is a single point. Therefore $SU(2)=S^3$. We could have got the same result in another way by recalling that the elements of $SU(2)$ can be represented as the matrices

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

where $|\alpha|^2+|\beta|^2=1$, α and β are complex numbers. The equation $|\alpha|^2+|\beta|^2=1$ defines the three-dimensional sphere S^3 in $\mathbf{C}^2$, indeed.

Further, $SU(2)$ is the group of quaternions of absolute value one, which is again S^3. The relation $SU(2)=S^3$ will be useful because the homology groups of S^3 are already known. For $n=3$ we have the fibration $SU(3) \xrightarrow{SU(2)} S^5$ i. e. $SU(3) \xrightarrow{S^3} S^5$.

By using the homology groups of the base and the fibre we are able to compute the term E_2 of the spectral sequence:

$$E_2^{p,q} = H_p(S^5, H_q(S^3))$$

	S^3								
4									
3	**Z**					**Z**			
2									
1									
0	**Z**					**Z**			S^5
	0	1	2	3	4	5	6		

The table immediately shows that $d_2 = d_3 = d_4 = \ldots = 0$ by dimensional considerations. Then $E_2 = E_\infty$, and in E_∞ each line $p+q=\text{const}$ contains at most one non-trivial group, i. e. the adjoint group is identical with the original one. (This follows from the third property of adjoint groups.)

Then for the homology groups of $SU(3)$ we have $H_0 = H_3 = H_5 = H_8 = \mathbf{Z}$ and $H_q = 0$ for the rest, i. e. $H_p(SU(3);\ \mathbf{Z}) = H_p(S^3 \times S^5;\ \mathbf{Z})$.

Consider the case $n=4$, i. e. the fibration

$$SU(4) \xrightarrow{SU(3)} S^7.$$

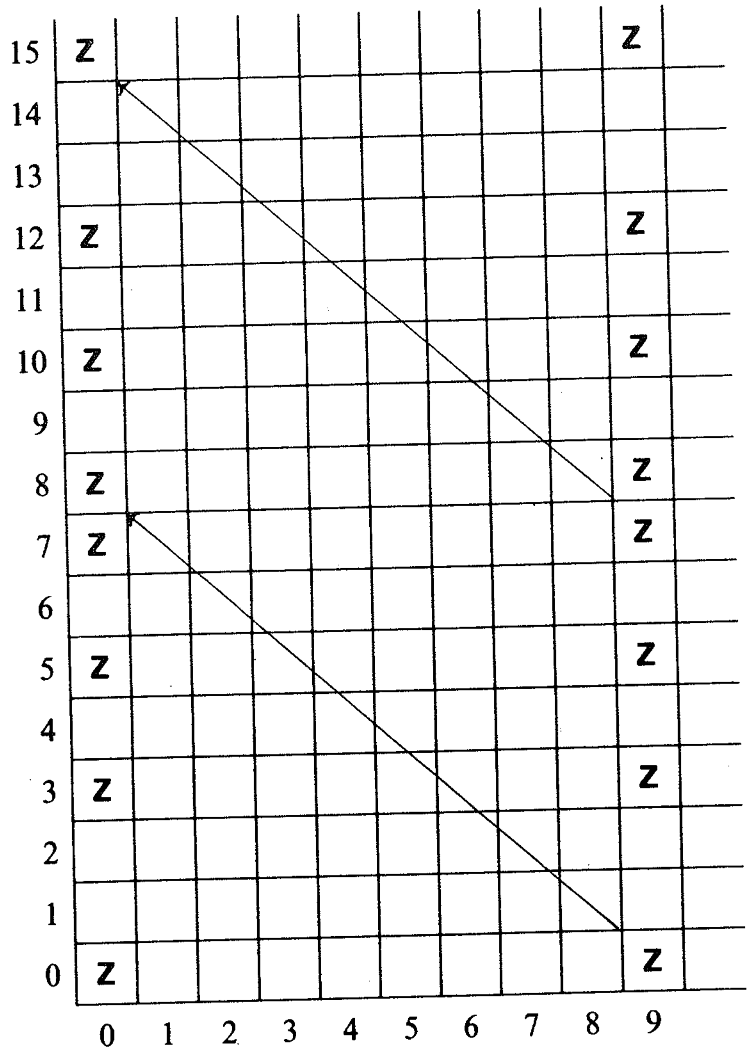
8	**Z**							**Z**	
7									
6									
5	**Z**							**Z**	
4									
3	**Z**							**Z**	
2									
1									
0	**Z**							**Z**	
	0	1	2	3	4	5	6	7	8

By consideration of the dimensions, $E_2 = E_3 = \ldots = E_\infty$ i. e. the spectral sequence is again trivial and $SU(4)$ has the homology groups $H_0 = H_3 = H_5 = H_7 = H_8 = H_{10} = H_{12} = H_{15} = \mathbf{Z}$ and for the rest $H_q = 0$. In other words

$$H_q(SU(4);\ \mathbf{Z}) = H_q(S^3 \times S^5 \times S^7;\ \mathbf{Z}).$$

One should not be led astray by the idea that this procedure can be carried on infinitely, by verifying step-by-step the triviality of the spectral sequence by dimension consideration. In fact, $n = 4$ turns out to be the last value of n for which simple consideration of the dimensions gives the full answer: for $n = 5$ further information is needed. Let us consider the fibration $SU(5) \xrightarrow{SU(4)} S^9$. The term E^2 is

By consideration of the dimensions we obtain $d_2 = d_3 = \ldots = d_8 = 0$ but d_9, as it seems, may be different from zero since $d_9^{9,0}\colon E_9^{9,0} \to E_9^{0,8}$, i. e. $d_9^{9,0}\colon \mathbf{Z} \to \mathbf{Z}$.



As a matter of fact d_9 is zero, yet we cannot prove it by merely using the facts at our disposal. (Here d_9 is the only "suspicious" differential since for $k>9$ all d_k are again zero by consideration of the dimensions.) Later on we shall prove the following:

Theorem.

$$H_*(SU(n); \mathbf{Z}) = H_*(S^3 \times S^5 \times \ldots \times S^{2n-1}; \mathbf{Z}).$$

We remark that for $n>2$ the spaces $SU(n)$ and $S^3 \times \ldots \times S^{2n-1}$ are not homeomorphic to each other, and they even have different homotopy groups (the reader may try to prove it, though it is not so simple).

In Chapter I, §9 we formulated the theorem of Freudenthal for suspensions: $\pi_i(X)=\pi_{i+1}(\Sigma X)$ for $i<2n-1$ where n is such a number that $\pi_0(X)=\pi_1(X)= \ldots =\pi_{n-1}(X)=0$.

Then it was proved only for the special case $X = S^n$. By applying the Leray theorem, we are now able to prove the general statement.

Remark. In the topology there exists a principle (the so-called Eckmann–Hilton duality) that establishes duality between, among others, the suspension and the loop space, the wedge and the direct product, the homotopy and the cohomology (we mentioned this in §2). Some examples for dual theorems:

$$\begin{cases} \pi_i(X)=\pi_{i-1}(\Omega X) & \text{for every } i, \\ H^i(X)=H^{i+1}(\Sigma X) & \text{for every } i; \end{cases}$$

$$\begin{cases} \pi_i(X\times Y)=\pi_i(X)+\pi_i(Y) & \text{for every } i, \\ H^i(X\vee Y)=H^i(X)+H^i(Y) & \text{for every } i; \end{cases}$$

$$\begin{cases} \pi_i(X)=\pi_{i+1}(\Sigma X) & \text{with some restrictions on } i, \\ H^i(X)=H^{i-1}(\Omega X) & \text{with some restrictions on } i. \end{cases}$$

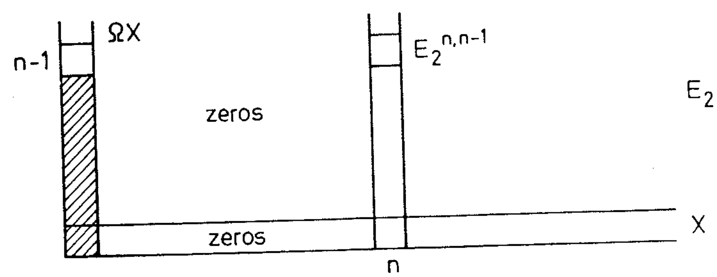
Let us now verify the last equality.

As it is known, for every space X there exists a Serre fibration whose space EX is contractible in itself to a single point and whose fibre is the loop space over X, i. e.

$$*\sim EX \xrightarrow{\Omega X} X.$$

Since X is $(n-1)$-connected, i. e. $\pi_0=\pi_1=\ldots=\pi_{n-1}=0$, we have $H^0(X)=H^1(X)=\ldots=H^{n-1}(X)=0$.

The spectral sequence is

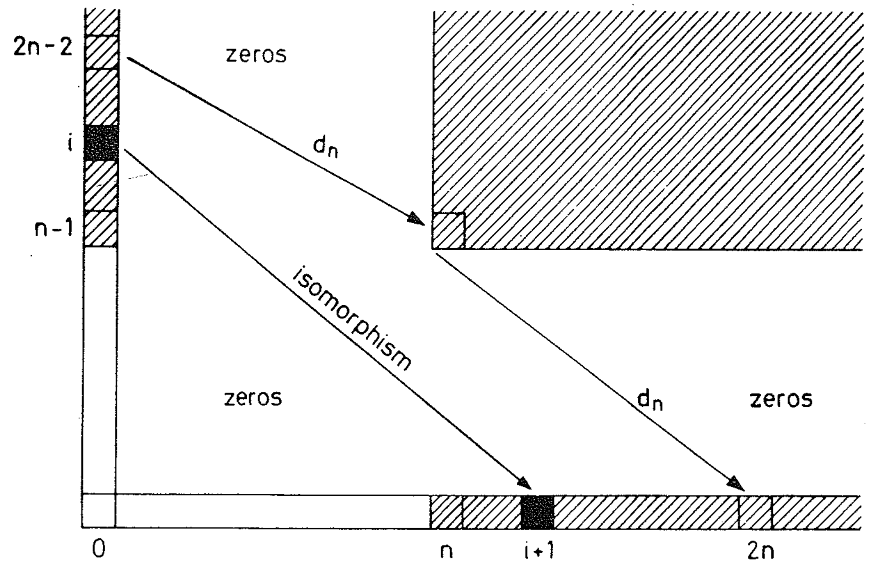


Now $EX\sim *$ and so E_∞ contains nothing but zeros. All differentials from the groups in the shaded column are equal to zero by dimension consideration. Therefore the column itself, being transferred by the differentials into the term E_∞ without modification, may contain only zero groups. Let us draw E_2 once more, shading the elements which can be different from zero. (See the next page.)

Let us follow the i-th group of the column on the left-hand side. If i is not very large, only d_{i+1} may be different from zero. As E_∞ is zero, d_{i+1} is an isomorphism (Ker $d_{i+1}^{i,0}=$ $=E_\infty^{i,0}$; Coker $d_{i+1}^{i,0}=E_\infty^{0,i+1}X$). Hence $H^i(\Omega X)=E_2^{i,0}=E_2^{0,i+1}=H^{i+1}(X)$.

We can even tell the largest i for which this observation holds. We must not forget about the "angle". How does it come into our considerations?

If $i\geq 2n-2$, $d_{i+1}^{0,i}$ is not any more the only non-trivial differential defined on $H^i(\Omega X)$. (For example, if $i=2n-2$, a differential $E_n^{0,2n-2}\to E_n^{n,n-1}$ is still possible.) Similarly, in



the case $i \geq 2n$ several non-trivial differentials are possible with values in $E_r^{i,0}$. In other words, at $i = 2n-3$ the differential d_{i+1} still slips by the angle but at $i = 2n-2$ it clings to it, therefore $H^i(X) = H^{i-1}(\Omega X)$ only for $i \leq 2n-2$.

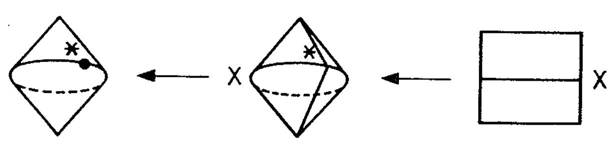
There exist a canonical imbedding i_X and a canonical projection π_X:

$$i_X: X \to \Omega \Sigma X; \quad \pi_X: \Sigma \Omega X \to X.$$

For the suspension we can choose between two slightly different definitions. Here we shall assume that

$$\Sigma X = X \times I/(X \times \{0\}) \cup (X \times \{1\}) \cup (* \times I)$$

where $*$ is the base point of X.



Let $x \in X$, then $i_X(x)$ must be a loop on the suspension; let $[i_X(x)](t) = (x, t)$. Further, set $\pi_X(\varphi, t) = \varphi(t)$ where t is a number, φ is a loop and $\varphi(t)$ a point in X.

Let us examine the homomorphism $\pi_X^*: H^i(X) \to H^i(\Sigma \Omega X)$. We know that $H^i(X)$ and $H^i(\Sigma \Omega X) = H^{i-1}(\Omega X)$ are isomorphic for $i < 2n-1$. As it will be proved in §21, the isomorphism is established by the particular mapping π_X^*. Let us consider the chain

$$\Sigma X \xrightarrow{\Sigma i_X} \Sigma \Omega \Sigma X \xrightarrow{\pi_{\Sigma X}} \Sigma X.$$

The first mapping is ordinary suspension over the mapping i_X while the second is π_Y where $Y = \Sigma X$. The composite $\pi_{\Sigma X} \circ \Sigma i_X: \Sigma X \to \Sigma X$ is identity. Really,

$$(x, t) \mapsto (i_X(x), t) \mapsto [i_X(x)](t) = (x, t).$$

If X is acyclical up to n, then so is ΣX up to $n+1$, therefore $\pi_{\Sigma X}$ induces isomorphism of the cohomology groups of ΣX and $\Sigma\Omega\Sigma X$ in the dimensions $\leq 2(n+1)-2 = 2n$. Hence the mapping $\Sigma i_X: \Sigma X \to \Sigma\Omega\Sigma X$ induces isomorphisms between the cohomology groups up to dimension $2n$ and the mapping i_X induces isomorphisms of the cohomology groups of X and $\Omega\Sigma X$ in the dimensions at most $2n$.

Next we are going to make use of the fact that i_X is simply an *imbedding* of X into $\Omega\Sigma X$.

We consider the exact cohomology sequence of the pair $(\Omega\Sigma X, X)$:

$$H^i(\Omega\Sigma X, X) \longrightarrow \underbrace{H^i(\Omega\Sigma X) \xrightarrow{i_X^*} H^i(X)}_{\text{(isomorphism for } i \leq 2n-1)} \longrightarrow H^{i+1}(\Omega\Sigma X, X)$$

This means that $H^i(\Omega\Sigma X, X) = 0$ for $i \leq 2n-1$.

Hence $H_i(\Omega\Sigma X, X) = 0$ if $i \leq 2n-2$. Now by the relative Hurewicz theorem the homotopy groups of the pair will be zero in the dimensions $\leq 2n-2$. (We assume $n > 2$, so reference to the theorem is justified.)

By employing the exact sequence of the pair we obtain: the inclusion mapping i_X: $X \to \Omega\Sigma X$ induces an isomorphism of the homotopy groups $\pi_i(X) = \pi_i(\Omega\Sigma X) = \pi_{i+1}(\Sigma X)$ for $i \leq 2n-2$. Q. e. d.

In some cases (for instance, in the proof of the theorem of H. Cartan in §28) we shall need the following addition to the Freudenthal theorem:

Theorem. In the critical dimension the suspension homomorphism Σ: $\pi_{2n-1}(X) \to \pi_{2n}(\Sigma X)$ is an *epimorphism.*

Proof. We recall the following theorem of Whitehead. Let X and Y be two arbitrary simply-connected spaces and let f: $X \to Y$ be such a mapping that f_*: $\pi_2(X) \to \pi_2(Y)$ is an epimorphism. Then the following two statements are equivalent:

(1) the homomorphism f_*: $\pi_m(X) \to \pi_m(Y)$ is an isomorphism for $m < n$ and an epimorphism for $m = n$;

(2) the homomorphism f_*: $H_m(X) \to H_m(Y)$ is an isomorphism for $m < n$ and an epimorphism for $m = n$. (The homologies are taken over $\mathbf{Z}$.)

Let us now turn to the original statement. We shall consider cohomology rather than homology spectral sequences. We can use the old picture, by simply turning the arrows in the opposite direction. Obviously $H_{2n-2}(X) = H_{2n-3}(\Omega X)$.

By substituting ΣX for X we get

$$H_{2n-1}(\Omega\Sigma X) = H_{2n}(\Sigma X) = H_{2n-1}(X).$$

Since this isomorphism is induced by the inclusion i: $X \to \Omega\Sigma X$, the homomorphism $\pi_{2n-1}(X) \to \pi_{2n-1}(\Omega\Sigma X) \to \pi_{2n}(\Sigma X)$ is an epimorphism by the Whitehead theorem. Q.e.d.

Remark 1. Here we have isomorphism between the $2n-1$-dimensional homology groups, which is more than what the Whitehead theorem requires. Nevertheless it does not help us to prove isomorphism rather than epimorphism between the corresponding homotopy groups. The Whitehead theorem implies only epimorphism, and nothing more.

Remark 2. One can actually do without referring to the Whitehead theorem. The equality $H_{2n-1}(X)=H_{2n-1}(\Omega\Sigma X)$ implies that $H_{2n-1}(\Omega\Sigma X;\ X)=0$ (all the preceding groups are known to be isomorphic). By applying the relative Hurewicz theorem one obtains

$$\pi_{2n}(\Omega\Sigma X,\ X)\to\pi_{2n-1}(X)\to\pi_{2n-1}(\Omega\Sigma X)\to\underset{\displaystyle 0}{\underset{\displaystyle \|}{\pi_{2n-1}(\Omega\Sigma X,\ X)}}\to\pi_{2n-2}(X)$$

Here $\pi_{2n}(\Omega\Sigma X, X)\neq 0$ in general. Therefore $\pi_{2n-1}(X)\to\pi_{2n-1}(\Omega\Sigma X) = \pi_{2n}(\Sigma X)$ is always epimorphism but not necessarily isomorphism.

§21. AN ADDENDUM TO THE LERAY THEOREM

Let X and Y be two spaces, both of them filtered, and let f be a mapping of X to Y compatible with the filtrations, i. e. $f(X_k)\subset Y_k$, $k=0,\ldots,n$.

Then f induces a homomorphism of the homology spectral sequences, i. e. homomorphisms ${}^XE_r^{p,q}\to{}^YE_r^{p,q}$ for every p, q and r. They commute with the differentials, and so all the properties of the groups commute with the homomorphisms. The same can be said about cohomology spectral sequences only the arrow must be in the opposite direction.

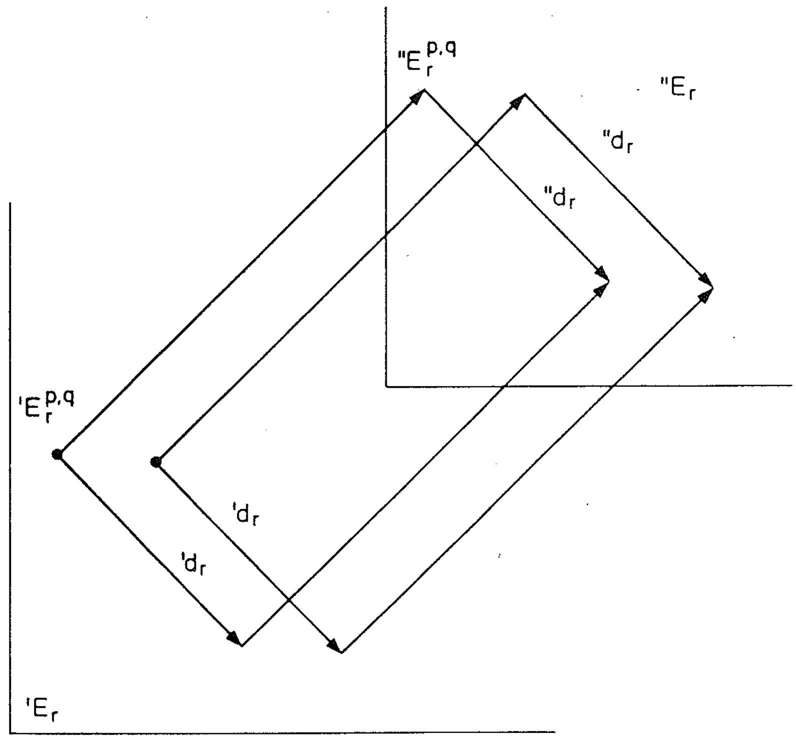
Assume now that we are given two fibrations (E_1, B_1, F_1, p_1) and (E_2, B_2, F_2, p_2) and a mapping of fibrations, i. e. a commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\tilde f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

The spaces B_1 and B_2 are assumed to be CW complexes (let us recall the remark made in the first paragraph of §19). The mapping is homotopic to a cellular one, so it can be considered as such, and $\tilde f: E_1\to E_2$ as a mapping compatible with the filtrations of X and Y.

That generates a homomorphism of the (homology) spectral sequences $'E_r^{p,q}\to''E_r^{p,q}$ (where $'$ and $''$ denotes that the item belongs to the first or second fibration, respectively). We have, among others, a mapping $'E_2^{p,q}\to''E_2^{p,q}$. Now f takes the fibre F_1 into the fibre F_2. As it can easily be seen, $'E_2^{p,q}\to''E_2^{p,q}$ coincides with the mapping $H_p(B_1, H_q(F_1))\to$ $\to H_p(B_2, \mathrm{H}_q(F_2))$ induced by f: $B_1\to B_2$ and $\tilde f|_{F_1}: F_1\to F_2$.

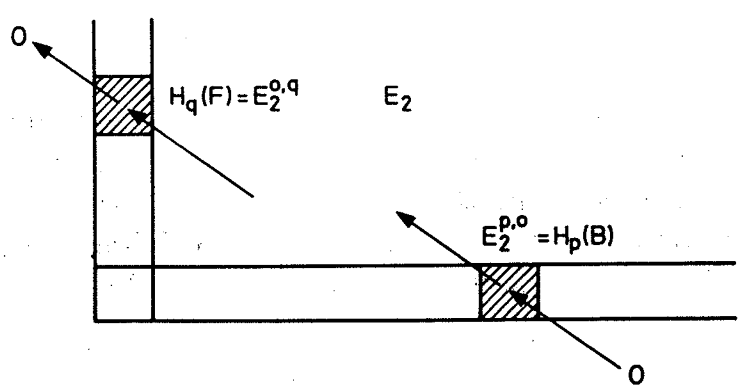
Since the homomorphism of the groups $E_r^{p,q}$ commutes with the action of the differentials, if the homomorphism $'E_k^{p,q} \to {}''E_k^{p,q}$ is an isomorphism for some particular $k=r$ for every p and q, the same is true for all $k \geq r$; the differentials $'d_k^{p,q}$ and $''d_k^{p,q}$ will act in the same way.



The spectral sequence is, up to isomorphism, independent of the particular cellular structure of the base.

Indeed, if it is somehow divided to cells, there exists a homotopy, connecting the identity mapping of the base with a cellular mapping of it into itself, which is a cellular mapping of the first cellular decomposition to the second. This homotopy induces isomorphism of $'E_2$ and $''E_2$ as has been shown above, and the isomorphism of $'E_r$ and $''E_r$ follows for every $r>2$.

The analogous statement is true for the cohomology spectral sequences (up to the direction of the arrows).



Let us now examine the homology spectral sequence of a Serre fibration $p\colon E \to B$.

The Leray theorem is usually formulated in more detail by adding three statements which give a good grasp of the general situation. The first one concerns the first column on the left-side; the second one concerns the bottom row; the third one informs about the connection between the left-side column and the bottom row.

The left column of E_2 consists of the groups $H_q(F)$. All elements of $H_2(F)$ are cycles with respect to the action of d_2; some of these elements are "covered" by elements coming out of the inside of the table, thus transition from $H_q(F) = E_2^{0,q}$ to $E_3^{0,q}$ is made by factorization, and so on. Each consecutive step is by factorization of the previous group. We come to an end at some group $E_\infty^{0,q} = {}_{(0)}H_q(E)/{}_{(-1)}H_q(E) = \operatorname{Im} H_q(E^0) \subset$ $\subset H_q(E)$, i. e. we have a chain of mappings:

$$H_q(F) = E_2^{0,q} \to E_2^{0,q}/\ldots \to E_3^{0,q}/\ldots \to \ldots \to E_\infty^{0,q} \subset H_q(E).$$

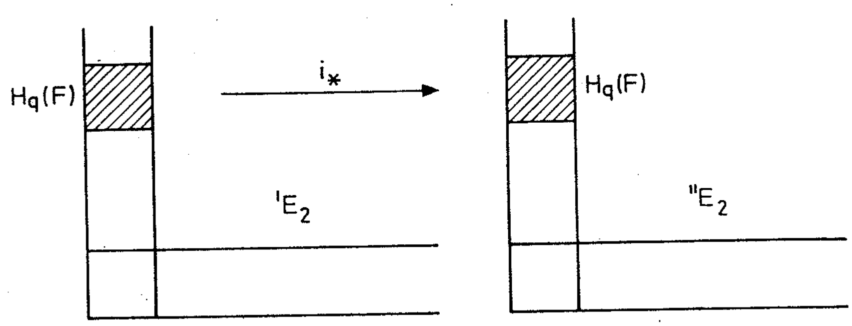
All the arrows are projections of groups to their quotient groups; in the last step we have imbedding, i. e. we have obtained a mapping $H_q(F) \to H_q(E)$.

Now there exists an imbedding $i\colon F \to E$ and the corresponding mapping of the homology groups. The *first addition* to the Leray theorem states that the mapping we have constructed is nothing else than the mapping i_* induced by the imbedding $i\colon F \to E$.

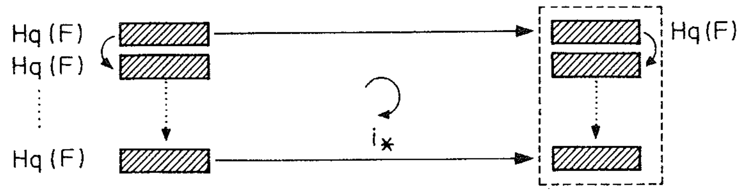
Proof. Let us consider the fibrations (E, B, F, p) and $(F, *, F, p)$, and the obvious imbedding

$$\begin{array}{ccc} F & \xrightarrow{\;i\;} & E \\ F\big\downarrow & & \big\downarrow F \\ * & \longrightarrow & B \end{array}$$

which induces a homomorphism of spectral sequences.



The sequence of $(F, *, F, p)$ is trivial and consists of a single column; the mapping i_* is induced by the imbedding $i\colon F \to E$. By passing from E_2 to E_3 and so on, we have the left-side table unaltered while the left-side column of the right-side table starts going through the factorization process considered. Once this is finished, we have on the left-side the same $H_q(F)$ as before while on the right-side the result of a chain of mappings.



The chain of mappings we are concerned with is framed by dotted line. The left column is an identical copy of the group $H_q(F)$. Now let us notice that in the second term the mapping $'E_2^{0,q} \to {}''E_2^{0,q}$ is an isomorphism and that the diagram is commutative. Then in the resulting square we have isomorphism on the upper and the left-side. The first statement is proved.

Let us consider the second addendum. We begin with examining the bottom row of E_2, i. e. the family $\{H_q(B)\}$. No element of $E_2^{q,0}$ can be image of a differential, therefore no factorization takes place as we are passing from $E_2^{q,0}$ to $E_3^{q,0}$ only "cleaning" i. e. ignoring all elements of $E_2^{q,0}$ which are not cycles (sent to zero by the differential d_2); in other words, transition from $E_2^{q,0}$ to $E_3^{q,0}$ means transition from the whole $H_q(B)$ to some subgroup, and so on. We obtain a chain of mappings:

$$H_q(B) = E_2^{q,0} \supset E_3^{q,0} \supset E_4^{q,0} \supset \dots \supset E_\infty^{q,0},$$

i. e. $E_\infty^{q,0} \subset H_q(B)$. On the other hand, $E_\infty^{q,0}$ is known to be a quotient group of $H_q(E)$, hence there is a natural mapping of $H_q(E)$ to $E_\infty^{q,0}$ and so, to $H_q(B)$. We have obtained a mapping $H_q(E) \to H_q(B)$. The *second additional statement* to the Leray theorem says that the obtained mapping is nothing else than the mapping of the homology groups induced by the projection $p\colon E \to B$.

Proof. The arguments repeat our former considerations. Again we consider two fibrations (E, B, F, p) and $(B, B, *, \pi)$ and the mapping of fibrations

$$\begin{array}{ccc} E & \xrightarrow{p} & B \\ F\big\downarrow p & \curvearrowright & *\big\downarrow \pi \\ B & \longrightarrow & B \end{array}$$

The spectral sequence of the second fibration consists of the first row alone which is not changed by transition from $''E_r$ to $''E_{r+1}$. Under $p_*\colon \dot{E}_2 \to {}''E_2$ the elements of the first row are being mapped isomorphically, then transition from $'E_2^{q,0}$ to $'E_3^{q,0}$ etc. starts the process of realizing the chain of mappings in question.

The rest of the proof is word-by-word the same as above. The second addendum is proved.

We have proved both statements for homology spectral sequences. Since the proofs are similar in the case of cohomology we limit its treatment to the formulation of the following statements:

The mappings

$$H^q(B) = E_2^{q,0} \to E_2^{q,0} / \bigoplus_{r \geq 2} \operatorname{Im} d_r = E_\infty^{q,0} = {}^{(q-1)}H^q(E) \subset H^q(E)$$

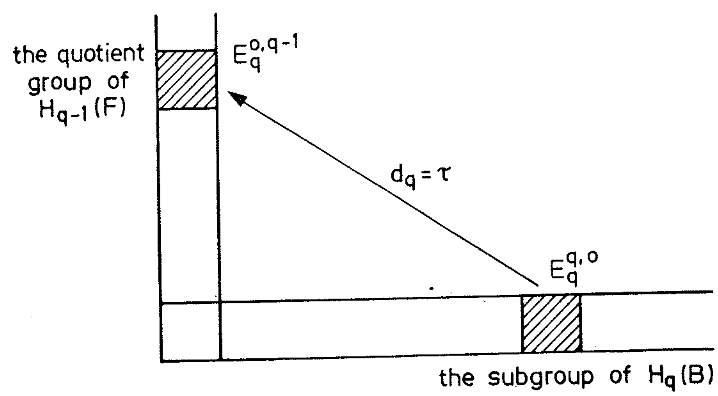
and $$H^q(E) \to H^q(E)/{}^{(0)}H^q(E) = E_\infty^{0,q} = \bigcap_{r \geq 2} \operatorname{Ker} d_r^{0,q} \subset E_2^{0,q} = H^q(F)$$

coincide with the mapping $p^*: H^q(B) \to H^q(E)$ and $i^*: H^q(E) \to H^q(F)$ induced by the projection of the fibration and the imbedding of the fibre, respectively.

The most interesting addendum is the third, especially as it includes the definition of transgression, a notion playing outstanding role in the theory of fibred spaces and in the cohomology theory of compact Lie groups.

The transgression

Let us examine the term E_2. As we were examining the behaviour of the first row and first column we observed that as r was growing the elements of the row were being "swept up" in the sense that $E_2^{q,0}$ lessened and was replaced by its subgroup; on the other hand, the group $E_2^{0,q}$ in the column was replaced by the quotient groups. That is, both groups were decreasing but in entirely different ways. Therefore, had we stopped at some moment at a certain r we should find a subgroup of $H_q(B)$ in the cell $(q, 0)$ and a quotient group of $H_{q-1}(F)$ in $(0, q-1)$. If we stopped at $r=q$ we would find that the appropriate differential d_q was acting just from the cell $(q, 0)$ to $(0, q-1)$.



The differential d_q: $E_q^{q,0} \to E_q^{0,q-1}$ will be called *transgression*, and denoted by τ.

Transgression is a partially-defined, multivalued mapping. Indeed, it is a mapping of a subgroup to a quotient group (thus being not everywhere defined and assigning to each element of its domain a whole coset). The elements of $H_q(B)$ that belong to $E_q^{q,0}$ are called transgressive.

Let us now consider a purely geometric construction.

The mapping of pairs p: $(E, F) \to (B, *)$ induces a mapping $H_q(E, F) \to H_q(B, *) =$

$= H_q(B)$. By means of the boundary homomorphism of the exact sequence of the pair (E, F) we obtain a diagram

$$H_q(B) = H_q(B,*) \xleftarrow{p_*} H_q(E, F) \xrightarrow{\partial} H_{q-1}(F)$$

$$\gamma = p_*^{-1}$$

(assuming that $q \geq 2$ and the base is simply connected). Then the third addendum to the Leray theorem states that $\partial \circ \gamma = \tau$.

The notion of transgression may be similarly formulated for the cohomological case. Then fibre and base change their places, the transgressive elements appear in cohomology and τ: [subgroup of $H^q(F)$] $\to$ [quotient group of $H^{q+1}(B)$].

We shall give the definition of transgression in cohomological terms by applying so-called "transgression cochains".

Let (E, B, F, p) be a fibration. The imbedding $i: F \to E$ induces an epimorphism of the cochains $\mathscr{C}^q(E)$ to the cochains $\mathscr{C}^q(F)$; $i^*: \mathscr{C}^q(E) \to \mathscr{C}^q(F)$. The mapping $p^*: \mathscr{C}^{q+1}(B) \to$ $\to \mathscr{C}^{q+1}(E)$ is a canonical monomorphism, thus $\mathscr{C}^{q+1}(B)$ is imbedded into $\mathscr{C}^{q+1}(E)$. The images in $\mathscr{C}^{q+1}(E)$ of the cochains $\mathscr{C}^{q+1}(B)$ will be called *basic cochains*. An element $z \in H^q(F)$ is *transgressive* if there exists a cochain $\alpha \in \mathscr{C}^q(E)$ such that $i^*(\alpha) \in z$ and $\delta\alpha$ is a basic cochain ($\delta\alpha$ is the coboundary of α). If $p^*(\omega) = \delta\alpha$, the class of the cochain ω (ω is a cocycle) will be called the image of the class z under the pretransgression $\hat{\tau}$, $\{\omega\}$ $= \hat{\tau}(z)$.

Even though $i^*(\alpha) \in Z^q(F)$ is a cocycle (α is called a *transgression cochain*) the cochain α is, as a rule, not a cocycle, i. e. $\delta\alpha \neq 0$ is general. The transgressive elements of $H^q(F)$ constitute a subgroup, thus $\hat{\tau}$ is defined on a subgroup, rather than on the whole $H^q(F)$. Further, the cocycle ω is not uniquely determined. Indeed, assume $p^*(\omega_1) = \delta\alpha_1$, $p^*(\omega_2) = \delta\alpha_2$ and $i^*(\alpha_1) = i^*(\alpha_2) \in z$. Then $\omega_1 - \omega_2 = \tilde{\omega}$ where $\tilde{\omega}$ is such that $p^*(\tilde{\omega}) = \delta\tilde{\alpha}$ where $i^*(\tilde{\alpha}) = 0$. The classes of the cocycles of the type $\tilde{\omega}$ form a subgroup $\Gamma^{q+1}(B) \subset$ $\subset H^{q+1}(B)$ and the image of an element z under $\hat{\tau}$ is defined in $H^{q+1}(B)$ up to elements from $\Gamma^{q+1}(B)$, thus one has a mapping τ: $\tau(z) = \hat{\tau}(z) \bmod \Gamma^{q+1}(B)$, i.e. $T^q(F) \to$ $\to H^{q+1}(B)/\Gamma^{q+1}(B)$ (here $T^q(F)$ denotes the set of transgressive elements of $H^q(F)$). The mapping τ is called a *transgression.*

This definition expresses, for short, that the transgression is the partially defined multivalued homomorphism of $H^{q-1}(F)$ to $H^q(B)$ given by the composition

$$H^{q-1}(F) \xrightarrow{\delta} H^q(E, F) \xrightarrow{(p^*)^{-1}} H^q(B).$$

Exercise. Prove that the "relative" definition is equivalent to that using "transgression cochains".

Exercise. Formulate the homological definition of transgression by applying "transgression chains".

Let us now prove the third additional statement to the theorem of Leray. Namely, we shall prove that the differential $d_q^{q,0}$ coincides with the transgression as defined in the homological case in terms of transgression chains. (We have not actually given the homological definition; anyhow, it is the exact analogue of the cohomological version.) Let us recall one of the definitions from the first section.

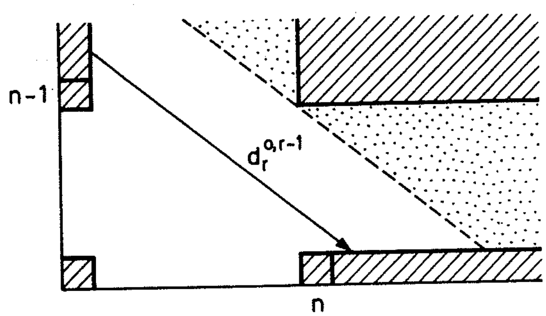
An element $\alpha \in E_0^{i,q-i} = \mathscr{C}_q(X_i)/\mathscr{C}_q(X_{i-1})$ belongs to $Z_r^{i,q-i} \subset E_0^{i,q-i}$ if and only if there exists a representative $a \in \mathscr{C}_q(X_i)$ of α whose boundary has a filtration smaller by r than has α, i. e. $\partial a \in \mathscr{C}_{q-1}(X_{i-r})$.

In the case of a fibration $X_i = p^{-1}(B^i)$ where B^i is the i-skeleton of the base space. Put $i=q, r=q$. Then $\mathscr{C}_{q-1}(X_{i-r}) = \mathscr{C}_q(p^{-1}(B^0)) = \mathscr{C}_{q-1}(F)$ (assume that the complex B has a single vertex; as earlier proved, this means no loss of generality) i. e. $\alpha \in Z_q^{q,0}$ if and only if α has a representative $a \in \mathscr{C}_q(E)$ such that $\partial a \in \mathscr{C}_{q-1}(F)$.

As for the differential, $d_q^{q,0}$ we have now the following. In the group $E_2^{q,0} = H_q(B)$ we have a subgroup $E_q^{q,0}$ consisting of of all elements $\alpha \in H_q(B)$ which are represented by cochains $a \in \mathscr{C}_q(B)$ whose pre-images $\tilde{a} \in \mathscr{C}_q(E)$ are such that $\partial \tilde{a} \in \mathscr{C}_{q-1}(F)$ (i. e. $\tilde{a} \in Z_q^{q,0}$). The last condition expresses that $\tilde{a}$ is a relative cycle of E mod F. The homology class of the cycle $\partial \tilde{a} \in \mathscr{C}_{q-1}(F)$ is one of the values of $\tau\alpha$ (by the definition of transgression) and also a representative of $d_q^{q,0}\alpha$ (by definition of the differential). Q.e.d. (Later on, we will rather often have proofs that contain only old definitions repeated quite a number of times.)

Let us stop at an important example where transgression has obvious geometric meaning.

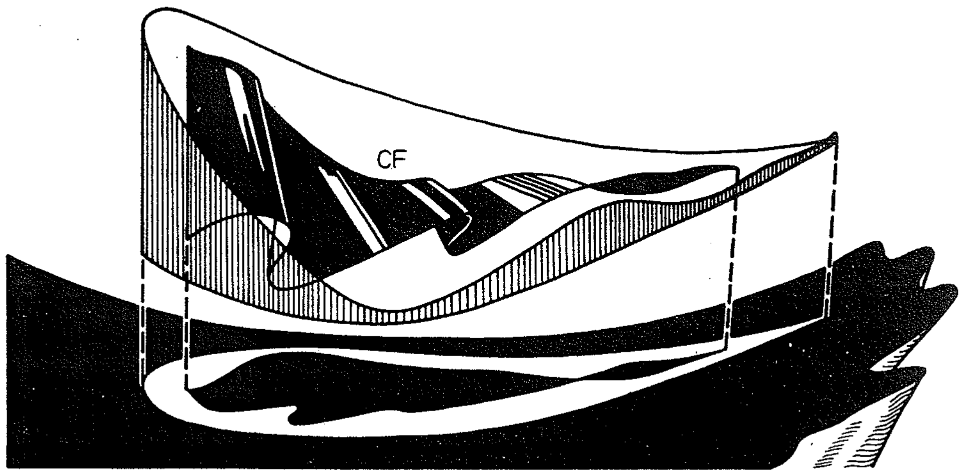
Let $\pi\colon E \xrightarrow{F} B$ be a fibration such that $\pi_1(B)=0$ and E is contractible. Let B be aspherical up to the dimension n, i. e. $\pi_i(B)=0$ for $i<n$. We consider the cohomological spectral sequence. By repeating the reasoning of §2 (where the fibration $EX \xrightarrow{\Omega X} X$ has been considered) we get the following picture:



The differential $d_r^{0,r-1}$ is an isomorphism as long as the angle has no effect on it. (We mean isomorphism between $H^i(F)$ and $H^{i+1}(B)$, for $d_r^{0,r-1}$ is an isomorphism for every sufficiently large r, however if $r > 2n-1$ the isomorphism will be between a subgroup of $H^{r-1}(F)$ and a quotient group of $H^r(B)$ rather than between $H^{r-1}(F)$ and $H^r(B)$.)

Now we are going to construct a mapping $H^r(B) \to H^{r-1}(F)$ which will be an isomorphism in the small dimensions ($r \leq 2n-1$) and the inverse of the transgression τ in all dimensions.

Since E is contractible, so is F in E, and the imbedding $F \to E$ may be extended to a mapping of the cone CF to E (this extension is, of course, not uniquely determined).



The projection π sends the bottom $F \times \{0\}$ of the cone into a single point, therefore a mapping $\Sigma F \to B$ arises. Let φ denote this mapping. It induces a homomorphism of the homology groups φ^*: $H^r(B) \to H^r(\Sigma F) \to H^{r-1}(F)$. The mapping is constructed for every value of r and is induced by a mapping of spaces. Moreover, for every r the transgression τ is defined and is a mapping (in this case an isomorphism) between some subgroup of $H^{r-1}(F)$ and a quotient group of $H^r(B)$. The inverse maps the quotient group of $H^r(B)$ onto the subgroup of $H^{r-1}(F)$, i. e. it can be considered as a (single-valued) homomorphism of $H^r(B)$ to $H^{r-1}(F)$. We want to show that it coincides with φ^* (that implies, in particular, that the homomorphism φ^* is independent of how φ has been constructed).

$H^{r-1}(F)$ and a quotient group of $H^r(B)$. The inverse maps the quotient group of $H^r(B)$ onto the subgroup of $H^{r-1}(F)$, i. e. it can be considered as a (single-valued) homomorphism of $H^r(B)$ to $H^{r-1}(F)$. We want to show that it coincides with φ^* (that implies, in particular, that the homomorphism φ^* is independent of how φ has been constructed).

If r is small both the transgression and φ^* are isomorphisms. In this case the meaning of the theorem is that they are the inverse mappings of each other. This was used in §19 in the particular case of the Serre fibration $EX \xrightarrow{\Omega X} X$. The mapping π_X: $\Sigma\Omega X \to X$ is one of the possible choices for φ (obtained by contracting EX to a point in the usual manner).

Thus the isomorphisms $H^q(X) \xrightarrow{\pi_X^*} H^q(\Sigma\Omega X) \xrightarrow{(\Sigma)^{-1}} H^{q-1}(\Omega X)$ and $(d_q^{0,q-1})^{-1}$ coincide. (We have repaid our debt together with all the interests.)

Let us prove the statement now.

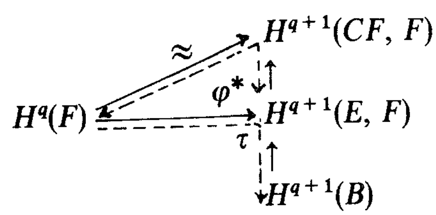
We construct a chain of mappings $(CF, F) \to (E, F) \to (B, *)$ where the first one is an imbedding and the second a projection. Consider the mapping of exact cohomology sequences induced by the former and examine a square in the diagram (it is commutative as all the others are):

$$\begin{array}{ccc} H^q(F) & \xrightarrow{\approx} & H^{q+1}(CF, F) \\ \uparrow & \curvearrowright & \uparrow \\ H^q(F) & \longrightarrow & H^{q+1}(E, F) \end{array}$$

Both rows of the square are isomorphisms. Let us consider the whole chain and the composite mapping

$$H^{q+1}(B) \to H^{q+1}(E, F) \to H^{q+1}(CF, F) \xrightarrow{(\approx)} H^{q+1}(\Sigma F) \xrightarrow{(\approx)} H^q(F)$$

which is the mapping φ^* in question. Let us collect the whole in one diagram:

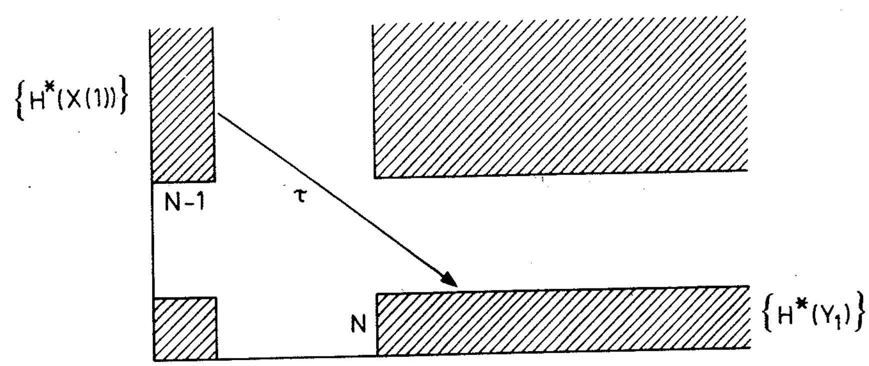


The dotted lines denote the transgression τ and the homomorphism φ^*. As the triangle is commutative, we have $\varphi^* \circ \tau \equiv 1_{H^q(F)}$, i. e. they are inverses to each other. Q.e.d.

The first obstruction to a section

We are going to show how to define and calculate the characteristic class of a fibration (i. e. the first obstruction to extending a section, cf. the end of §17).

Consider the cohomology spectral sequence of the fibration $E \xrightarrow{F} B$ with coefficients in $\pi_n(F)$ ($\pi_i(F) = 0$ if $i \leq n-1$). The term E_2 is as follows:



The base B is assumed to be simply connected. (This assumption comes from the obstruction theory where it has has been essential.) The first non-trivial group in the column of E_2 is $H^n(F; \pi_n(F))$. The fibre F is $(n-1)$-connected, therefore there exists in $H^n(F; \pi_n(F))$ a canonically distinguished element: every cell of dimension n is an n-dimensional sphere defining an element of $\pi_n(F)$; that means, there is defined an n-dimensional cochain with coefficients in $\pi_n(F)$; this cochain is actually a cocycle. (We have already used this construction but only as applied to the space $K(\pi; n)$; the fact that the cochain of $C^n(F, \pi_n(F))$ is a cocycle can be verified by repeating the proof that the cochain E (in $K(\pi, n)$) is a cocycle.)

The class of the cocycle E will be denoted by e. It is the same that we called earlier the fundamental cohomology class:

$$e \in H^n(F; \pi_n(F)).$$

The element e belongs to $E_2^{0,n}$; it cannot be the image of a differential, and since there is a trivial stripe in E_2 consisting of 1-st, ..., $(n-1)$-st rows, e is a cocycle with respect to the differentials $d_2, d_3, d_4, \ldots, d_n$. Hence it is transgressive, i. e. it belongs to $E_{n+1}^{0,n}$. The transgression $\tau = d_{n+1}$ maps e onto $\tau(e) \in H^{n+1}(B; \pi_n(F))$. This element is the characteristic class of the fibration.

We shall prove this with certain restrictions.

Let B be a simply-connected CW complex with a single vertex, E a CW complex and $p: E \to B$ a cellular mapping. Moreover the pre-image $p^{-1}(\sigma)$ of each cell $\sigma \subset B$ will be assumed to consist of a union of whole cells of E (i. e. if any cell of E intersects $p^{-1}(\sigma)$ it is contained in $p^{-1}(\sigma)$). The last assumption concerns the n-skeleton of E, which will be supposed to consist of the n-skeleton of $F \subset E$ (where F is the pre-image of the single vertex of B by p) and of a section over the n-skeleton of the base B.

These restrictions can be overcome by showing that every Serre fibration is homotopy equivalent to a fibration with the properties required. The reader may try to prove this, even though to prove this in general is more difficult than for any of the paricular cases we shall meet.

Let us consider a representative

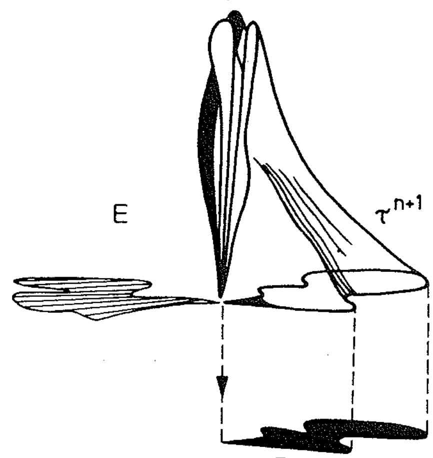
$$\bar{e} \in \mathscr{C}^n(F; \pi_n(F))$$

of the homology class $e \in H^n(F; \pi_n(F))$ and the cochain $c \in \mathscr{C}^{n+1}(B; \pi_n(F))$ defined by the section which was given over B^n. Then the group $\mathscr{C}^{n+1}(E; \pi_n(F))$ contains the cochains $\delta\bar{e}$ and p^*c. We show that $\delta\bar{e} = p^*c$.

Let τ^{n+1} be a cell of the complex E. As it follows from the assumptions τ^{n+1} is either projected to a cell σ^{n+1} of the base or to a cell of smaller dimension. The boundary $\partial\tau^{n+1}$ is a sum $\alpha_1 + \alpha_2$ where $\alpha_1 \in \mathscr{C}_n(F)$ and $\alpha_2 = p^*(\partial\sigma^{n+1})$ (if τ^{n+1} is projected to a cell of dimension smaller than $n+1$, we have $\alpha_2 = 0$).

Clearly $\delta\bar{e}(\tau^{n+1}) = \bar{e}(\alpha_1)$ i. e. we have the homotopy class of the chain α_1 (chains in $\mathscr{C}_n(F)$ are linear combinations of n-dimensional cells of F, which are spheres).

Further, we have $p^* c(\tau^{n+1}) = c(p_* \tau^{n+1}) = c(\sigma^{n+1})$, i. e. the class defined in $\pi_n(F)$ by the section over the boundary of σ^{n+1} (i. e. α_2). This projection defines a spheroid homotopic to α_1: the homotopy is realized by the image of the cell τ^{n+1}. Hence $\delta\bar{e} = p^* c$. Q. e. d.



§22. MULTIPLICATION IN COHOMOLOGY SPECTRAL SEQUENCES

Thus far we have used both homology and cohomology spectral sequences without experiencing any significant difference between them except that the arrows are directed opposite. The reason is clear: we never used the multiplicative structure of the cohomology. In what follows we shall concentrate on cohomology sequences.

Assume that the group of coefficients is a ring. (For example, a field, or **Z**.) Then the spectral sequence is equipped with a multiplicative structure.

Actually, for every $r \geq 2$ the group $E_r = \oplus_{p,q} E_r^{p,q}$ may be equipped with a homogeneous multiplication (i. e. there exists a mapping $E_r^{p,q} \otimes E_r^{p',q'} \to E_r^{p+p',q+q'}$) consistent with the differentials:

$$d(a \cdot b) = da \cdot b + (-1)^{p+q} a \cdot db$$

for any $a \in E_r^{p,q}$, $b \in E_r^{p',q'}$. Certainly, multiplication in E_{r+1} is induced by the multiplication given in E_r.

Multiplication in the spectral sequence will be compatible with that defined in E_2 and E_∞ by virtue of the Leray theorem. Let this be formulated more exactly, and in more detail. Consider the group $E_2 = \bigoplus_{p,q} E_2^{p,q}$; we have $E_2^{p,q} = H^p(B; H^q(F))$. By the above, E_2 is a ring. On the other hand $\oplus_{p,q} H^p(B; H^q(F)) = \oplus_p H^p(B; \oplus_q H^q(F))$ is a ring, too, for $\bigoplus_q H^q(F)$ is a ring and so is $\bigoplus_p H^p(B; \bigoplus_q H^q(F))$. We assert that the ring E_2 is isomorphic to $\bigoplus_p H^q(B; \bigoplus_q H^q(F))$.

(The same is true for any ring of coefficients.)

Consider now the subring $\oplus_p E_2^{p,0}$ of E_2. The theorem we have formulated implies the following: the rings $\oplus_p E_2^{p,0}$ and $H^*(B)$ are isomorphic. Similarly the rings $\oplus_q E_2^{0,q}$ and $H^*(F)$ are isomorphic, too.

Recall the formula of universal coefficients.

$$0 \to H^p(B;\ \mathbf{Z}) \otimes H^q(F;\ \mathbf{Z}) \to H^p(B;\ H^q(F;\ \mathbf{Z})) \to \mathrm{Tor}\,(H^{p+1}(B;\ \mathbf{Z});\ \ H^q(F,\ \mathbf{Z})) \to 0$$

The E_2 term obviously contains a subgroup isomorphic to $H^p(B; \mathbf{Z}) \otimes H^q(F; \mathbf{Z})$. If the cohomology groups of either the fibre or the base are torsion-free, then $\mathrm{Tor}=0$ and $H^p(B; H^q(F)) = H^p(B) \otimes H^q(F)$.

The same is obviously true if the coefficients are taken in a field. So we have $E_2 = H^*(B;\ K) \otimes H^*(F;\ K)$ whenever K is a field or $K = \mathbf{Z}$ and the base or the fibre is torsion-free. The multiplication in E_2 is then given by the formula

$$(a'\ b') \circ (a'' \otimes b'') = (-1)^{\dim b' \cdot \dim a''}(a' a'' \otimes b' b'').$$

The E_2 term we shall deal with, will usually be given as a tensor product. (Even in the cases when $H^p(B) \otimes H^q(F)$ does not coincide with E_2 it is a subgroup as well as a subring of E_2 and multiplication is given by the same formula.) For instance, this was the case for the unitary groups, in the example examined in §20.

Let us now consider the E_∞ term. Whenever a ring A is equipped with a filtration compatible with multiplication, i. e. $A_p \cdot A_q \subset A_{p+q}$, the adjoint group gets a ring structure; indeed, if $\alpha \in A_p/A_{p+1}$ and $\beta \in A_q/A_{q+1}$, an element of $A_{p+q}/A_{p+q+1} \subset GA$ may be assigned to them in the following way. We take representatives $a \in \alpha$ and $b \in \beta$ from A_p and A_q, respectively. The product ab lies in A_{p+q}; its representative in A_{p+q}/A_{p+q+1} depends on α and β alone and it is denoted by $\alpha \cdot \beta$.

As it turns out, the filtration

$$\ldots \subset {}^{(2)}H^*(E) \subset {}^{(1)}H^*(E) \subset {}^{(0)}H^*(E) \subset {}^{(-1)}H^*(E) = H^*(E)$$

is compatible with the multiplication in $H^*(E)$ and the multiplication given here in the adjoint group E_∞ coincides with that obtained in E_∞ by transition to E_∞ from the multiplication in E_2.

Remark. The multiplication in the adjoint ring is always somewhat poorer than in the original ring. Let $a, b \in E_\infty$; by the construction of $E_\infty^{p,q}$ these are families (cosets) of elements from $H^*(E)$. Suppose that $a \cdot b = c$ and $c \neq 0$ in E_∞. It follows then that the family of elements of $H^*(E)$ corresponding to c contains no null element, i. e. if multiplication is not trivial in E_∞, neither is it in $H^*(E)$, i. e. the amount of information that can be obtained on $H^*(E)$ is significant in this case. If however $a \cdot b = 0$ while $a \neq 0$ and $b \neq 0$, then the class $a \cdot b$ contains a representative belonging to a coset of *higher* filtration than supposed (by at least one unit), i. e. the product in $H^*(E)$ may be different from zero. In other words, triviality of multiplication in E_∞ does not imply the same in $H^*(E)$. In consequence, the information obtained about $H^*(E)$ is not complete.

Multiplication in the spectral sequence is constructed in the following way. Recall that if X and Y are two spaces and

$$c_1 \in \mathscr{C}^k(X), \quad c_2 \in \mathscr{C}^l(Y),$$

the tensor product

$$c_1 \otimes c_2 \in \mathscr{C}^{k+1}(X \times Y)$$

is defined and the equality

$$\delta(c_1 \otimes c_2) = \delta c_1 \otimes c_2 + (-1)^k c_1 \otimes \delta c_2$$

is valid. By transition to cohomology we get a multiplication

$$H^k(X) \otimes H^l(Y) \to H^{k+1}(X \times Y).$$

By using the diagonal mapping $\Delta\colon X \to X \times X$ a binary operation is introduced in $H^*(X)$ (namely, $\alpha \cdot \beta = \Delta^*(\alpha \otimes \beta)$). Tensor product is defined by the formula

$$(c_1 \otimes c_2)(f) = c_1(\pi_X f|_{A_0,\dots,A_k}) c_2(\pi_Y f|_{A_k,\dots,A_{k+l}})$$

where $f\colon (A_0,\dots,A_{k+l}) \to X \times Y$ is a singular simplex while $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ are projections. Here multiplication is taken in the sense of the ring structure of the domain of coefficients.

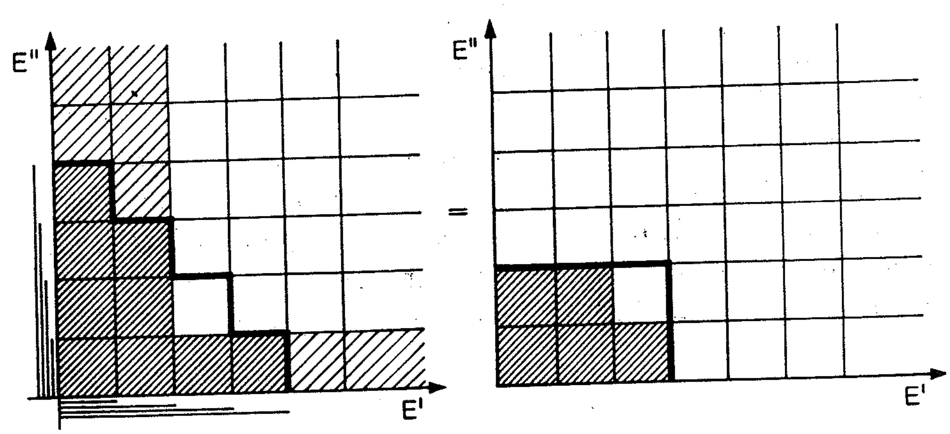
The analogue of this operation is defined on relative chains. If $c_1 \in C^k(X, X_1)$ and $c_2 \in \mathscr{C}^l(Y, Y_1)$, we have

$$c_1 \otimes c_2 \in \mathscr{C}^{k+l}(X \times Y, X_1 \times Y \cup X \times Y_1).$$

Multiplication in spectral sequences of fibrations will be defined from the beginning for products of two different fibrations, too.

Let (E', B', F', p') and (E'', B'', F'', p'') be fibrations and let $(E' \times E'', B' \times B'', F' \times p' \times \times p'')$ be their product. Let $\{(E')_p\}$, $\{(E'')_p\}$ and $\{(E' \times E'')_p\}$ denote the filtrations given by the pre-images of the skeleta of the bases.

Denote by $\{'E_r^{p,q}, 'd_r^{p,q}\}$, $\{''E_r^{p,q}, ''d_r^{p,q}\}$ and $\{E_r^{p,q}, d_r^{p,q}\}$ the cohomology spectral sequences of the fibrations (E', B', F', p'), (E'', B'', F'', p'') and $(E' \times E'', B' \times B''$,



$F' \times F''$, $p' \times p''$), respectively. Let $\alpha_1 \in' E^{p_1, q_1} = H^{p_1+q_1}((E')^{p_1}, (E')^{p_1-1})$ and $\alpha_2 \in'' E^{p_2,q_2} = H^{p_2+q_2}((E'')^{p_2}, (E'')^{p_2-1})$. The tensor product is

$$\alpha_1 \otimes \alpha_2 \in H^{p_1+q_1+p_2+q_2}((E')^{p_1} \times (E'')^{p_2}, (E')^{p_1} \times (E'')^{p_2-1} \times (E'')^{p_2}) =$$

(in view of the excision theorem)

$$= H^{p_1+q_1+p_2+q_2}((E' \times E'')^{p_1+p_2}, (E' \times (E'')^{p_2-1} \cup (E')^{p_1-1} \times E'') \cap (E' \times E'')^{p_1+p_2}.$$

Here we have a cohomology group modulo a space larger than $(E' \times E'')^{p_1+p_2-1}$, therefore it is mapped naturally to

$$H^{p_1+q_1+p_2+q_2}((E' \times E'')^{p_1+p_2}, (E' \times E'')^{p_1+p_2-1}) = E_1^{p_1+p_2, q_1+q_2}.$$

The image of $\alpha_1 \otimes \alpha_2$ in this group will also be denoted by $\alpha_1 \otimes \alpha_2$.

As it can be verified,

$$d_1^{p_1+p_2, q_1+q_2}(\alpha_1 \otimes \alpha_2) = 'd^{p_1,q_1}\alpha_1 \otimes \alpha_2 + (-1)^{p_1+q_1}\alpha_1 \otimes {}''d_1^{p_2,q_2}\alpha_2.$$

By virtue of this formula, the multiplication $'E_1^{p_1,q_1} \otimes {}''E_1^{p_2,q_2} \to E_1^{p_1+p_2,q_1+q_2}$ defines a multiplication $'E_2^{p_1,q_1} \otimes {}''E_2^{p_2,q_2} \to E_2^{p_1+p_2,q_1+q_2}$ which, in turn, defines another one in E_3, then one in E_4, and so on.

Thus, multiplication is defined for every r, p_1, q_1, p_2, q_2, assigning to each pair $\alpha_1 \in {}'E_r^{p_1,q_1}$, $\alpha_2 \in {}''E_r^{p_2,q_2}$ some $\alpha_1 \otimes \alpha_2 \in E_2^{p_1+p_2,q_1+q_2}$.

Finally, the diagonal mapping of fibrations

$$\begin{array}{ccc} E & \xrightarrow{\Delta} & E \times E \\ p \downarrow & & \downarrow p \times p \\ B & \xrightarrow{\Delta} & B \times B \end{array}$$

is applied to define multiplication in the spectral sequence of a single fibration (E, B, F, p) by choosing for product of $\alpha_1 {}^P E_r^{p,q_1}$ and $\alpha_2 {}^P E_r^{p_2,q_2}$ the element $\alpha_1\alpha_2 = \Delta^*(\alpha_1 \otimes \alpha_2)$ where Δ^* is the homomorphism of spectral sequences induced by the diagonal mapping of fibrations.

The verification of all the properties of the multiplication, listed above in this section, will be left to the reader with the warning that it will be a laborious, though rewarding, work.

Let us see a good example for the use of the multiplicative structure.

We have not yet finished the study of the cohomology of the unitary group $SU(n)$. We already have the integral homology of $SU(n)$ if $n = 2, 3, 4$:

$$H_*(SU(n); \mathbf{Z}) = H_*(S^3 \times S^5 \times \ldots \times S^{2n-1}; \mathbf{Z}).$$

Hence, in view of $U(n) = S^1 \times SU(n)$ we have

$$H_*(U(n); \mathbf{Z}) = H_*(S^1 \times S^3 \times \ldots \times S^{2n-1}; \mathbf{Z})$$

where $n = 1, 2, 3, 4$.

It will be shown that similar equality holds for every n and for homology and cohomology alike. In cohomology the equality means ring isomorphism, i. e.

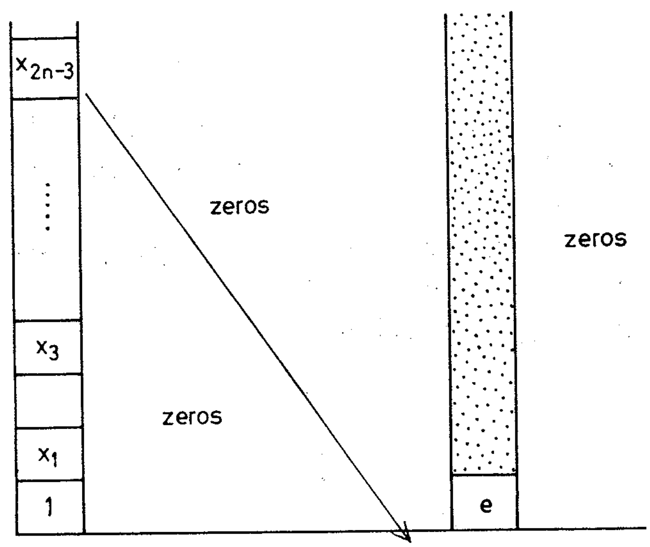
$$H^*(U(n), \mathbf{Z}) = \Lambda(x_1, x_3, x_5, \ldots, x_{2n-1})$$

where on the right we have the exterior algebra generated by $\{x_i\}$ where $\deg x_i = i$. It will be clear when and how the difficulties, experienced with $SU(5)$, will be overcome.

If $n=2$, the theorem is proved. Suppose that it is true for every $k \le n-1$, i. e.

$$H^*(U(n-1); \mathbf{Z}) = \Lambda(x_1, x_3, \ldots, x_{2n-3}).$$

Consider the fibration $U(n) \xrightarrow{U(n-1)} S^{2n-1}$. In its cohomology spectral sequence E_2 looks as follows.



The reason why we could not go on was that we met a differential that was not trivial because of simple reasons of dimension. At the second attempt we realize that the "dubious" differential is acting from a cell that contains, instead of one of the generators of the algebra $\Lambda(x_1, x_3, \ldots, x_{2n-3})$, an element composed of them. If we only consider the generating elements, we see that all the differentials under consideration happily pass below the dangerous cell $E_2^{2n-1,0}$, and so all are trivial. So are the differentials of the products.

We obtained that all the differentials are trivial, i. e. $E_2 = E_\infty$. Thus

$$E_\infty = H^*(S^{2n-1}; H^*(U(n-1); \mathbf{Z})) = H^*(S^{2n-1}; \mathbf{Z}) \otimes H^*(U(n-1); \mathbf{Z}),$$

where equality means ring isomorphism, i. e.

$$G(H^*(U(n); \mathbf{Z}) = H^*(S^{2n-1}; \mathbf{Z}) \otimes H^*(U(n-1); \mathbf{Z}).$$

Every diagonal $p+q = $ const contains no more than two non-zero (free) groups; hence $E^{n_2, q-n_2} = H^q(E; \mathbf{Z}) / {}^{(n_1)}H^q(E; \mathbf{Z})$, where $n_1 > n_2$.

As we have free groups we obtain $H^*(U(n); \mathbf{Z})$ by taking the direct sums over the diagonals $p+q = \text{const}$, i. e.

$$H^*(U(n); \mathbf{Z}) = \wedge(x_1, x_3, \ldots, x_{2n-1}) = H^*(S^{2n-1} \times U(n-1); \mathbf{Z})$$

in the sense of additive isomorphism.

Actually this is also a ring isomorphism for on every diagonal we have factorization by the group which is in the second column ${}^{(n_1)}H^q(E; \mathbf{Z})$. Now the whole second column is obtained from the first one by tensor multiplying with the generator $e = x_{2n-1}$, i. e. it only consists of products. The ring isomorphism $E_\infty = H^*(U(n))$ follows from the remark concerning adjoint rings in §22.

Exercise. Determine the cohomology ring of the complex and the quaternion Stiefel manifold.

The cohomology rings of projective spaces

(1) $H^*(\mathbf{CP}^n;\ \mathbf{Z})$.

Consider the fibration $\pi: S^{2n+1} \to \mathbf{CP}^n$ the fibre of which is a circle (the projection π assigns to each point $(z_0, \ldots, z_n) \in S^{2n+1}$ the point $(z_0 : \ldots : z_n) \in \mathbf{CP}^n$). As $\mathbf{CP}^n$ is simply connected the term E_2 of the spectral sequence of the fibration is of the following form:

	zeros				
1	cohomology of $\mathbf{CP}^n$	0	0	0	...
0	cohomology of $\mathbf{CP}^n$	0	0	0	...
	$0\ \ 1 \ldots \qquad \ldots 2n$				

By consideration of dimension we have

$$E_3 = E_4 = \ldots = E_\infty = H^*(S^{2n+1}),$$

hence $E_3^{p,q} = 0$ for all (p, q) except $(0, 0)$ and $(2n, 1)$, and $E_3^{2n,1} = \mathbf{Z}$; in consequence $E_2^{1,0} = 0$, $E_2^{2n,1} = \mathbf{Z}$ and the differential $d_2^{k,1}$: $E_2^{k,1} \to E_2^{k+2,0}$ is an isomorphism if $k = 0, 1, \ldots, 2n-1$. Now $E_2^{k,0} = E_2^{k,1}$, therefore $E_2^{k,0} = E_2^{k,1} = \mathbf{Z}$ for $k = 0, 2, \ldots, 2n$ and the remaining $E_2^{k,0}$ are trivial. Hence $H^k(\mathbf{CP}^n; \mathbf{Z}) = \mathbf{Z}$ for $k = 0, 2, \ldots, 2n$ and $H^k(\mathbf{CP}^n; \mathbf{Z}) = 0$, otherwise as we already know. Then the E_2 term is as follows

e_0e	0	e_1e	0	e_2e	0	...	...	0	e_ne	0
e_0	0	e_1	0	e_2	0	...	...	0	e_n	0
0	1	2	3	4	5			$2n-1$	$2n$	

where $e_i \in H^{2i}(\mathbf{CP}^n;\ \mathbf{Z})$, $e \in H^1(S^1;\ \mathbf{Z})$ are the generators. They may be chosen so that $d_2^{2k,1}(e_k e) = e_{k+1}$ ($d_2^{2k,1}$ is an isomorphism!). Then $e_{k+1} = d_2^{2k,1}(e_k e) = e_k d_2^{0,1} e = e_k e_1$, hence $e_k = e_1^k$ for $k = 1, 2, \ldots, n$.

We have obtained that $H^*(\mathbf{CP}^n;\ \mathbf{Z}) = \mathbf{Z}[e_1]/\{e_1^{n+1}\}$ and $\dim e_1 = 2$.

Similarly the same holds for every ring A, $H^*(\mathbf{CP}^n;\ A) = A[e_1]/\{e_1^{n+1}\}$. For the infinite dimensional projective space we have $H^*(\mathbf{CP}^\infty;\ A) = A[e_1]$, in particular, $H^*(\mathbf{CP}^\infty;\ \mathbf{Z}) = \mathbf{Z}[e_1]$.

(2) $H^*(\mathbf{RP}^n;\ \mathbf{Z})$.

The additive structure is already known: if n is even,

$$H^q(\mathbf{RP}^n;\ \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{for } q = 0, \\ \mathbf{Z}_2 & \text{for } q = 2, 4, \ldots, n, \\ 0 & \text{for all other } q. \end{cases}$$

if n is odd,

$$H^q(\mathbf{RP}^n;\ \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{for } q = 0, n, \\ \mathbf{Z}_2 & \text{for } q = 2, 4, \ldots, n-1, \\ 0 & \text{for all other } q. \end{cases}$$

If n is odd ($n = 2k+1$) multiplication in $\mathbf{RP}^n$ is defined from the mapping $\pi_k\colon \mathbf{RP}^{2k+1} \to \mathbf{CP}^k$. (The mapping $\pi\colon S^{2k+1} \to \mathbf{CP}^k$ considered above, sends diametrically opposing points to the same point, and so it can be written as the composition morphisms

$$\pi_k^*\colon \mathbf{Z} = H^q(\mathbf{CP}^k;\ \mathbf{Z}) \twoheadrightarrow H^q(\mathbf{RP}^{2k+1};\ \mathbf{Z}) = \mathbf{Z}_2$$

(as easily verified in view of the cell construction of $\mathbf{CP}^k$ and $\mathbf{RP}^{2k+1}$).

One has therefore the following relations between generators $\bar{e}_i \in H^{2i}(\mathbf{RP}^{2k+1};\ \mathbf{Z})$: $\bar{e}_1^{k'} = \bar{e}_{k'}$, if $k' \le k$. Hence

$$H^*(\mathbf{RP}^{2k+1};\ \mathbf{Z}) = \mathbf{Z}_2[\bar{e}_1]/\{\bar{e}_1^k\} \otimes \wedge [f]$$

where $f \in H^{2k+1}(\mathbf{RP}^{2k+1};\ \mathbf{Z})$ is the canonical generator. The inclusion $\mathbf{RP}^{2k} \subset \mathbf{RP}^{2k+1}$ obviously induces isomorphism of the cohomology groups in all dimensions up to $2k$, therefore

$$H^*(\mathbf{RP}^{2k};\ \mathbf{Z}) = \mathbf{Z}_2[\bar{e}_1]/\{\bar{e}_1^{k+1}\}.$$

For the infinite-dimensional projective space we have

$$H^*(\mathbf{RP}^\infty;\ \mathbf{Z}) = \mathbf{Z}_2[\bar{e}_1].$$

(3) $H^*(\mathbf{RP}^n;\ \mathbf{Z}_2)$.

Again the additive structure is known:

$$H^q(\mathbf{RP}^n;\ \mathbf{Z}_2)=\begin{cases}\mathbf{Z}_2 & \text{if } q\le n,\\ 0 & \text{if } q>n.\end{cases}$$

As for the multiplicative structure, examine the mapping π_k: $\mathbf{RP}^{2k+1}\to\mathbf{CP}^k$ which is obviously a fibration with the circle as its fibre. The E_2 term of its spectral sequence mod 2 is of the following form:

zeros

ε	0	$\varepsilon\varepsilon_1$	0	$\varepsilon\varepsilon_1^2$	0	...	...	0	$\varepsilon\varepsilon_1^k$	0	
1	0	ε_1	0	ε_1^2	0	...	...	0	ε_1^k	0	
0	1	2	3	4	...			$2k-1$	$2k$		

Hence we conclude, taking into account the groups $H^q(\mathbf{RP}^n;\mathbf{Z}_2)$, that all the differentials of the sequence are trivial, $E_2 = E_\infty$ and, in consequence, there are the following relations between the generators α_i of $H^i(\mathbf{RP}^n;\mathbf{Z}_2)$: $\alpha_2^s = \alpha_{2s}$ and $\alpha_1\alpha_{2s} = \alpha_{2s+1}$. There remains one question we cannot answer yet: what is α_1^2 equal to? There are two possibilities: either $\alpha_1^2=\alpha_2$ which implies $\alpha_s=\alpha_1^s$, or $\alpha_1^2=0$. The spectral sequence alone will not give us the answer.

Actually we have $\alpha_1^2 = \alpha_2$ but that has to be proved by some *ad hoc* considerations. If we knew the necessary preliminaries we could apply notions of differential topology (Poincaré duality, intersection of cycles) and have a simple proof; now as we are, we rather go back to the original definition of multiplication in cohomology.

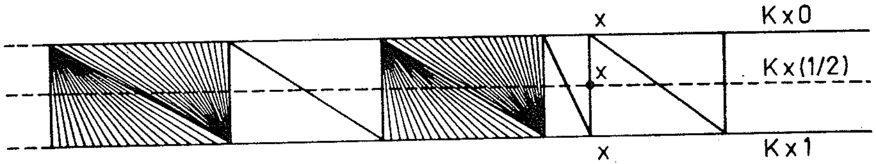
Let K be a finite simplicial complex with enumerated vertices $a_1, a_2, \dots, a_N$ and let $c_1\in\mathscr{C}^p(K;\ A)$ and $c_2\in\mathscr{C}^q(K;\ A)$ be cochains, i. e. functions defined on the p resp. q dimensional simplexes of K with values in the ring A. Their product $c_1c_2\in\mathscr{C}^{p+q}(K;\ A)$ is defined by the equality

$$c_1c_2(a_{i_0},\dots,a_{i_{p+q}})=c_1(a_{i_0},\dots,a_{i_p})c_2(a_{i_{p+1}},\dots,a_{i_{p+q}}),$$

where $(a_{i_0}, a_{i_1}, \dots, a_{i_m})$ is the simplex with the vertices $a_{i_0}, \dots, a_{i_m}$.

If c_1 and c_2 are cocycles in the cohomology classes γ_1 and γ_2 then c_1c_2 is a cocycle in the cohomology class $\gamma_1\gamma_2$. Indeed, by definition $\gamma_1\gamma_2=\Delta^*(\gamma_1\otimes\gamma_2)$ where Δ: $K\to K\times K$ is the diagonal imbedding, and $\gamma_1\otimes\gamma_2$ is the class of the cocycle $c_1\otimes c_2$, taking on $\sigma\times\tau$ the value $c_1(\sigma)c_2(\tau)$ if dim $\sigma=p$ and dim $\tau=q$ and 0 otherwise. Let us now construct a cellular approximation of the mapping Δ (we notice that $K\times K$ is a CW complex but not a simplicial one). Take the product $K\times[0,\ 1]$ and divide it to simplexes in the usual way (product $(a_{i_0}, \dots, a_{i_m})\times[0,\ 1]$ is divided to $m+1$ $(m+1)$-dimensional simplexes with vertices $(a_{i_0}\times 0,\dots,a_{i_k}\times 0,\ a_{i_k}\times 1,\dots,a_{i_m}\times 1)$, $k=0,\ 1,$

$\ldots, m$). In each simplex two opposite faces are chosen: $(a_{i_0}\times 0, \ldots, a_{i_k}\times 0)$ and $(a_{i_k}\times 1, \ldots, a_{i_m}\times 1)$. The line segments connecting the points of one segment with those of the other will not cross each other and will fill the whole simplex. Let us consider these segments in all simplexes of $K\times I$; they cover the whole complex.



Let $x\in K$. Consider the segment passing through the point $(x, 1/2)$. Let its endpoints be denoted by $(\varphi_0(x),0)$ and $(\varphi_1(x),1)$. The mapping $\tilde{\Delta}$: $K\to K\times K$ defined by the formula $\tilde{\Delta}(x)=(\varphi_0(x),\varphi_1(x))$ is an approximation of the diagonal imbedding (homotopy follows from the fact that if $x\in\sigma$ then $\tilde{\Delta}(x)\in\sigma\times\sigma$ and so it can be connected with $\Delta(x)$ by a segment) and is cellular; moreover $\tilde{\Delta}$ maps a simplex $(a_{i_0},a_{i_1},\ldots,a_{i_m})\subset\subset K$ onto the union of the products

$$(a_{i_0})\times(a_{i_0}, a_{i_1}, a_{i_2}, \ldots, a_{i_m})$$
$$(a_{i_0}, a_{i_1})\times(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$$
$$(a_{i_0}, a_{i_1}, a_{i_2})\times(a_{i_2}, \ldots, a_{i_m})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(a_{i_0}, a_{i_1}, a_{i_2}, \ldots, a_{i_m})\times(a_{i_m})$$

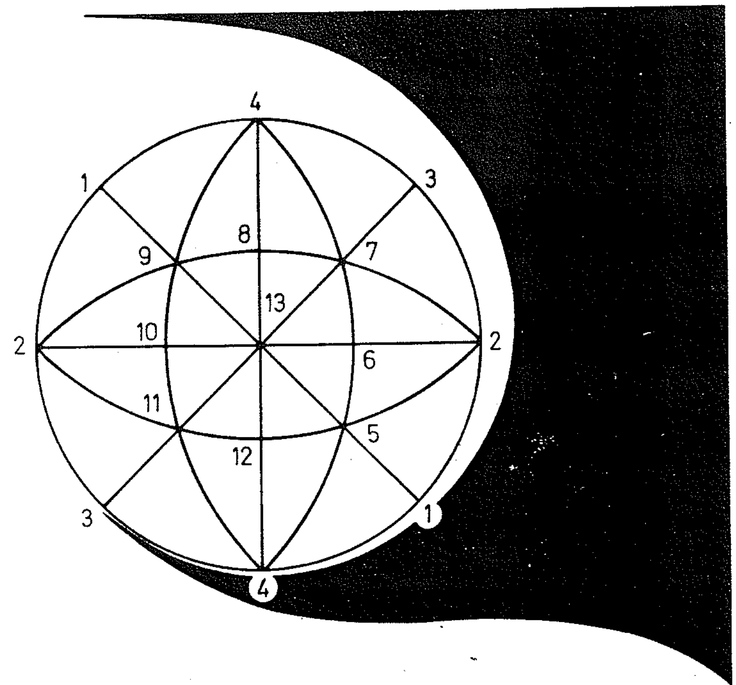
homeomorphically and with the orientation preserved.

The product $\gamma_1\gamma_2$ is the cohomology class of the cocycle $\tilde{\Delta}^*(c_1\otimes c_2)$. We have

$$[\tilde{\Delta}^*(c_1\otimes c_2)](a_{i_0}, a_{i_1},\ldots,a_{i_{p+q}})=(c_1\otimes c_2)(\tilde{\Delta}(a_{i_0}, a_{i_1}, \ldots, a_{i_{p+q}}))=$$
$$=\sum_{k=0}^{p+q}(c_1\otimes c_2)((a_{i_0}, \ldots, a_{i_k})\times(a_{i_k}, \ldots, a_{i_{p+q}}))=$$
$$=c_1(a_{i_0}, a_{i_1},\ldots,a_{i_p})c_2(a_{i_p},\ldots,a_{i_{p+q}}) = c_1(a_{i_0}, a_{i_1},\ldots, a_{i_p})c_2(a_{i_p},\ldots, a_{i_{p+q}}).$$

Hence $\tilde{\Delta}^*(c_1\otimes c_2)=c_1c_2$ i. e. the cocycle c_1c_2 belongs to the class $\gamma_1\gamma_2$ and that was to be shown.

Let us now consider $H^*(\mathbf{RP}^n; \mathbf{Z})$. It is sufficient to examine the case $n=2$ (as there is an imbedding $\mathbf{RP}^2\subset\mathbf{RP}^n$ for every n, which induces isomorphism of the cohomology groups in dimensions 1 and 2). The projective plane may be divided into 24 simplexes with 13 vertices. The one-dimensional cochain which assigns $1\in\mathbf{Z}_2$ to the simplexes 12, 14, 15, 19, 23, 25, 26, 27, and $0\in\mathbf{Z}_2$ to the rest, is a cocycle (verify it!) not cohomological to zero (its scalar product with the cycle $12+23+34+41$, which is not cohomological to zero, is equal to 1). We have



$$a^2(1\ 2\ 5) = a(1\ 2)\,a(2\ 5) = 1$$
$$a^2(1\ 2\ 9) = a(1\ 2)\,a(2\ 9) = 0$$
$$a^2(1\ 4\ 5) = a(1\ 4)\,a(4\ 5) = 0$$
$$a^2(1\ 4\ 9) = a(1\ 4)\,a(4\ 9) = 0$$
$$a^2(2\ 3\ 7) = a(2\ 3)\,a(3\ 7) = 0$$
$$a^2(2\ 3\ 11) = a(2\ 3)\,a(3\ 11) = 0$$
$$a^2(2\ 5\ 6) = a(2\ 5)\,a(5\ 6) = 0$$
$$a^2(2\ 6\ 7) = a(2\ 6)\,a(6\ 7) = 0$$
$$a^2(2\ 9\ 10) = a(2\ 9)\,a(9\ 10) = 0$$
$$a^2(2\ 10\ 11) = a(2\ 10)\,a(10\ 11) = 0$$
$$a^2(3\ 4\ 7) = a(3\ 4)\,a(4\ 7) = 0$$
$$a^2(3\ 4\ 11) = a(3\ 4)\,a(4\ 11) = 0$$
$$a^2(4\ 5\ 12) = a(4\ 5)\,a(5\ 12) = 0$$
$$a^2(4\ 7\ 8) = a(4\ 7)\,a(7\ 8) = 0$$
$$a^2(4\ 8\ 9) = a(4\ 8)\,a(8\ 9) = 0$$
$$a^2(4\ 11\ 12) = a(4\ 11)\,a(11\ 12) = 0$$
$$a^2(5\ 6\ 13) = a(5\ 6)\,a(6\ 13) = 0$$
$$a^2(5\ 12\ 13) = a(5\ 12)\,a(12\ 13) = 0$$
$$a^2(6\ 7\ 13) = a(6\ 7)\,a(7\ 13) = 0$$
$$a^2(7\ 8\ 13) = a(7\ 8)\,a(8\ 13) = 0$$
$$a^2(8\ 9\ 13) = a(8\ 9)\,a(9\ 13) = 0$$
$$a^2(9\ 10\ 13) = a(9\ 10)\,a(10\ 13) = 0$$
$$a^2(10\ 11\ 13) = a(10\ 11)\,a(11\ 13) = 0$$
$$a^2(11\ 12\ 13) = a(11\ 12)\,a(12\ 13) = 0$$

(Here (mn) and (mnp) denote the simplexes with vertices with indexes m, n, and m, n, p respectively.

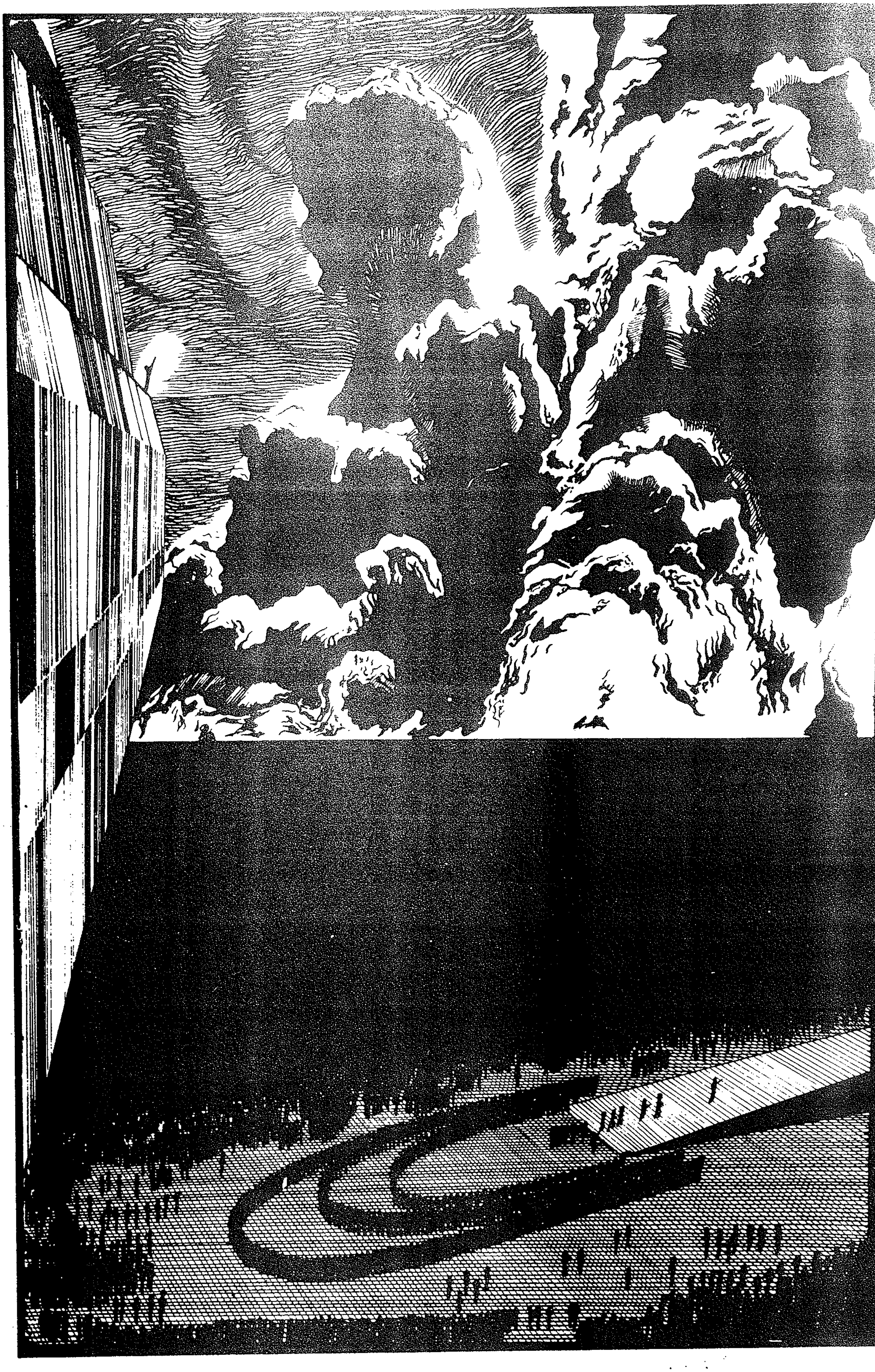
Thus the value of a^2 taken on the generator of the group $H_2(\mathbf{RP}^2; \mathbf{Z})$ is equal to $1 \in \mathbf{Z}_2$ (this generator is represented by the sum of all simplexes), therefore $a^2 \sim 0$.

So it has been shown that in the cohomology of $\mathbf{RP}^n$ mod 2, the square of the generator is the two-dimensional generator and

$$H^*(\mathbf{RP}^n; \mathbf{Z}_2) = \mathbf{Z}_2[\alpha_1]/\{\alpha_1^{n+1}\}$$

for every n. For the infinite-dimensional projective space we have

$$H^*(\mathbf{RP}^\infty, \mathbf{Z}_2) = \mathbf{Z}_2^{[\alpha_1]}$$



§23. KILLING SPACES

Let us see now how to calculate homotopy groups of topological spaces.

Let us be given a topological space X and suppose that our task is to find the homotopy groups $\pi_i(X)$ while $H^*(X)$ is assumed to be known. In the main we are interested in the case $\pi_1(X) = 0$. If $\pi_i(X) = 0$ for $i<n$, then by the Hurewicz theorem $\pi_n(X)=H_n(X;\ \mathbf{Z})$. To determine the other homotopy groups we shall use a clever geometric method. Once the integer cohomology groups of X are known, so is its cohomology with coefficients in an arbitrary Abelian group π. Let $\pi=\pi_n(X)=H_n(X;\ \mathbf{Z})$ We are going to construct a mapping

$$f\colon X\to K(\pi,\ n) = K(\pi_n(X),\ n).$$

One procedure is that homotopy groups are "glued" together, beginning from $\pi_{n+1}(X)$. By the addition theorem in §10, the group $\pi_n(X)$ will not change while X is becoming a subspace of a space of type $K(\pi,\ n)$.

There is an alternative procedure. We take the fundamental class e in $H^*(X,\pi_n(X))$. Since $H^*(X;\ \pi)=\Pi(X;\ K(\pi,\ n))$, the class e gives rise to a well-defined mapping $f\colon X\to K(\pi,\ n)$.

Both methods give some mapping of X to $K(\pi,\ n)$ that induces isomorphism between $\pi_n(X)$ and $\pi_n(K(\pi,\ n)) = \pi_n(X)$.

We know that every continuous mapping $X\to K(v_n(x),\ n)$ can be replaced by a homotopy equivalent fibration. The fibre will be denoted by $X|_n$ and called a killing space for X.

We can show still another procedure, though it is actually only a variant of the first. Consider the following Serre fibration:

$$*\sim E\xrightarrow{\ \Omega K(\pi,\ n)\ }K(\pi,\ n).$$

Obviously $\Omega K(\pi,n)=K(\pi,n-1)$. We construct over X the fibration induced by $f\colon X\to K(\pi,\ n)$.

$$\begin{array}{ccc} X|_n & \xrightarrow{\ \tilde f\ } & E\sim * \\ K(\pi_n(X),\ n-1)\ \Big\downarrow & & \Big\downarrow\ K(\pi_n(X),\ n-1 \\ X & \xrightarrow{\ f\ } & K(\pi_n(X),\ n) \end{array}$$

The space $\tilde X|_n$ of the induced fibration turns out to have the same homotopy type as $X|_n$. Indeed, if f is a fibration (as it can be assumed) then so if $\tilde f$, even having the same fibre. Now $E\sim *$, so the fibre of this last fibration must have homotopy type of $\tilde X|_n$. Then $\tilde X|_n\approx X|_n$.

Calculate the homotopy groups of $X|_n$. Consider the fibration:

$$f\colon X\xrightarrow{\ X|_n\ } K(\pi_n(X),\ n).$$

(In the sequel we shall often use the notation K_n rather than $K(\pi, n)$ whenever it causes no confusion.) The homotopy sequence is

$$\ldots \to \pi_i(K_n) \to \pi_{i-1}(X|_n) \to \pi_{i-1}(X) \to \pi_{i-1}(K_n) \to \ldots$$

Let $i \neq n$ and $i \neq n+1$. Then

$$\pi_i(K_n) = \pi_{i-1}(K_n) = 0,$$

and hence

$$\pi_{i-1}(X|_n) = \pi_{i-1}(X).$$

Thus $\pi_k(X|_n) = \pi_k(X)$ for $k \neq n,\ n-1$.

$$0 \to \pi_n(X|_n) \to \pi_n(X) \xrightarrow{\alpha} \pi_n(K_n) \to \pi_{n-1}(X|_n) \to \pi_{n-1}(X) = 0.$$

Consider the homomorphism α. Let it be recalled that $f\colon X \to K_n$ induces an isomorphism of the n-th homotopy groups, i. e. $\alpha = f_*$ is an isomorphism. Then $\pi_{n-1}(X|_n) = \pi_n(X|_n) = 0$. Thus

$$\pi_i(X|_n) = \begin{cases} 0 & \text{for } i \leq n, \\ \pi_i(X) & \text{for } i \geq n+1. \end{cases}$$

The first nontrivial homotopy group of the space X is killed. Suppose that the cohomology of $K(\pi, n)$ is known. Then, using the spectral sequence of any of the above fibrations, we can try to find the cohomology $H^*(X|_n)$, then $H_*(X|_n)$, and then in view of the Hurewicz theorem we have $\pi_{n+1}(X) = \pi_{n+1}(X|_n) = H_{n+1}(X|_n)$. Further the same procedure is repeated, that time with $X|_n$ instead of X, and so a new killing space $X|_{n+1}$ will have been obtained, and so on.

This means that once the cohomology of $K(\pi, n)$ is known we are able to compute the homotopy groups of an arbitrary topological space X. (How far this procedure is from its actual realization will be clear soon.)

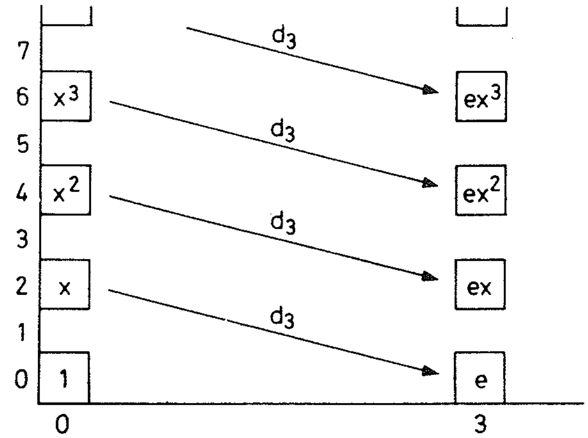
Let us illustrate the method of killing spaces on the following elementary problem: we shall compute $\pi_4(S^3)$.

$$X = S^3;\ \pi_1(X) = \pi_2(X) = 0,\ \pi_3(X) = \mathbf{Z};\ n = 3$$

$$X|_3 \xrightarrow{K(\mathbf{Z},\,2)} X; \quad K(\pi_n(X);\ n-1) = K(\mathbf{Z};\ 2),$$

and, as we already know, $K(\mathbf{Z}, 2) = \mathbf{CP}^\infty$, i. e. we have $X|_3 \xrightarrow{\mathbf{CP}^\infty} S^3$. The cohomology of $\mathbf{CP}^\infty$ is well known: $H^*(\mathbf{CP}^\infty, \mathbf{Z}) = \mathbf{Z}[x]$ where $\deg x = 2$ ($\mathbf{Z}[x]$ is the ring of polynomials of the generator x).

Let us examine the spectral sequence. The E_2 term is



We have $d_2=0$, i. e. $E_2=E_3$; $d_k=0$ for all $k<3$, i. e. $E_4=E_5=\ldots=E_\infty$; $d_3(x)\in H^3(S^3, \mathbf{Z})$. What is the value $d_3(x)$ equal to? Clearly $d_3(x)=\pm e$. Indeed, suppose $d_3(x)=ke$ (where $k\in\mathbf{Z}$ and is not necessarily different from zero). Then for every $k\neq\pm 1$ in E_∞ there will remain some nontrivial groups on at least one of the diagonals $p+q=2$ and $p+q=3$. Therefore at least one of the groups will be different from zero. This implies that $H^2(X|_3;\ \mathbf{Z})$ or $H^3(X|_3;\ \mathbf{Z})$ is different from zero which contradicts that $\pi_i(X|_3)=0$ for $i\leq 3$.

Thus we have $d_3(x)=\pm e$, hence $d_3(x^k)=kx^{k-1}d_3(x)=\pm kx^{k-1}e$ ($\deg x$ is even, thus d_3 acts with the same rules as the ordinary differential) i. e. $E_\infty^{3,2k}\cong\mathbf{Z}_{k+1}$ and, in particular, $E_\infty^{3,2}\cong\mathbf{Z}_2$. Because $E_\infty^{3,2k}$ are on the odd diagonals $2k+3=p+q$ while $E_\infty^{0,2s}$ are on the even ones, there is in E^∞ at most one nontrivial group on every diagonal; therefore no nontrivial adjointness arises, and $H^*(X|_3;\mathbf{Z})=E_\infty$. In particular, $H^5(X|_3;\mathbf{Z})=E_\infty^{3,2}=\mathbf{Z}_2$, i. e. $H_4(X|_3;\mathbf{Z})=\mathbf{Z}_2$. Hence $\pi_4(S^3|_3)=\pi_4(S^3)=\mathbf{Z}_2$.

This and the Freudenthal theorem imply that $\pi_{n+1}(S^n)=\mathbf{Z}_2$ for $n\geq 3$.

§24. THE RANKS OF THE HOMOTOPY GROUPS

As we have seen in the last section, in order to compute homotopy groups, in the first place we must know the cohomology of the spaces $K(\pi, n)$.

This task will prove far from easy.

It is relatively easy to compute the cohomology of these spaces with coefficients in the rational number field $\mathbf{Q}$. Obviously this information cannot satisfy our needs but at least gives something. It helps us to find the ranks of the homotopy groups of a space X, i. e. to find the groups $\pi_i(X)\otimes\mathbf{Q}$.

And so, let π be a group with finitely many generators. We shall compute $H^*(K(\pi, n);\ \mathbf{Q})$.

Because $K(\pi_1\oplus\pi_2, n)=K(\pi_1, n)\times K(\pi_2, n)$, it is sufficient to compute $H^*(K(\pi, n);\ \mathbf{Q})$ in the case when π is a finite periodical group, or $\pi=\mathbf{Z}$.

Theorem. (1) If π is a finitely generated group then $H^i(K(\pi, n); \mathbf{Q}) = 0$ for every $i > 0$ and $n > 0$.

(2) If n is odd then $H^i(K(\mathbf{Z}, n); \mathbf{Q}) = \mathbf{Q}$ for $i = 0$, $i = n$ and $H^i(K(\mathbf{Z}, n); \mathbf{Q}) = 0$ for all other i, and the square of the generator $e_n \in H^n(K(\mathbf{Z}, n); \mathbf{Q})$ in the cohomology ring is zero (i. e. $H^*(K(\mathbf{Z}, n); \mathbf{Q}) = \Lambda_{\mathbf{Q}}(e_n)$ is the exterior algebra with the single generator e_n).

(3) If n is even then $H^i(\mathbf{Z}, n); \mathbf{Q}) = \mathbf{Q}$ for $i = kn$, $k = 0, 1, 2, \ldots$ and $H^i(K(\mathbf{Z}, n); \mathbf{Q}) = 0$ for all other i, and the k-th power e_n^k of the generator $e_n \in H^n(K(\mathbf{Z}, n); \mathbf{Q})$ is a generator in $H^{kn}(K(\mathbf{Z}, n); \mathbf{Q})$ (i. e. $H^*(K(\mathbf{Z}, n); \mathbf{Q}) = \mathbf{Q}[e_n]$ is the polynomial ring over $\mathbf{Q}$ with a single generator e_n).

Proof. First of all we prove a lemma.

Lemma. Any two spaces of type $K(\pi, n)$ are weakly homotopy equivalent.

Indeed, let X and Y be two $K(\pi, n)$ spaces; Y is arbitrary while X is supposed to be that particular one we have constructed in §10. (We recall that X is a CW complex having a single vertex and no cells in dimensions from 1 to $n-1$, whose n-dimensional and $(n+1)$-dimensional cells are in one-to-one correspondence with the generators of π and the relations of the generators of π, respectively.)

Let us now construct $f\colon X \to Y$, a mapping that induces an isomorphism of the homotopy groups. this is exactly what we need because it will imply that X and Y are weakly homotopy equivalent and so are any two spaces of type $K(\pi, n)$.

A mapping f on the n-skeleton of X will be defined as follows. Each n-dimensional cell of X is a generator of $\pi = \pi_n(Y)$. The closure of such a cell is an n-dimensional sphere. Let the vertex of X be mapped to the base point of Y and let each sphere $\bar{\sigma}_i^n \subset X$ be mapped by means of the mapping $S^n \to Y$ representing the class $\sigma_i^n \in \pi = \pi_n(Y)$.

We now have a mapping on the n-skeleton of X. The obstruction to its extension to the $(n+1)$-skeleton is in $\mathscr{C}^{n+1}(X; \pi_n(Y))$. It assigns to a cell $\sigma^{n+1} \subset X$ the element of $\pi_n(Y)$ given as the restriction pf f to $\dot{\sigma}^{n+1}$. Now the boundary of σ^{n+1} in X is $\Sigma_i a_i \sigma_i^n$ and the element $\Sigma_i a_i \sigma_i^n$ is zero in π (as the very relation that made the cell σ^{n+1} attached to the complex). Therefore the restriction of f to $\dot{\sigma}^{n+1}$ defines the null element in $\pi_n(Y) = \pi$ and the mapping may be extended to the $(n+1)$-skeleton.

Further extension meets obtruction only in zero groups ($\pi_i(Y) = 0$ for $i > n$) and is therefore possible.

Thus we have obtained a mapping $f: X \to Y$ that induces an isomorphism between $\pi_n(X)$ and $\pi_n(Y)$ (by construction) and between $\pi_i(X)$ and $\pi_i(Y)$ for $i \neq n$ (because these are trivial groups). The lemma is proved.

It follows from the lemma that for given π and n, the cohomology rings of any $K(\pi, n)$ spaces are the same. We may therefore construct $K(\pi, n)$, in the course of the proof of the theorem, in any particular way. (We may assume, for instance, that $K(\pi, n) = \Omega K(\pi, n+1)$.)

Let us now return to the theorem. For $n = 1$ the statement is true. Indeed, as we may choose the circle for $K(\mathbf{Z}, 1)$ we get

$$H^i(K(\mathbf{Z}, 1); \mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{for } i = 0, 1, \\ 0 & \text{for } i > 1. \end{cases}$$

For $K(\mathbf{Z}_m, 1)$ we may take the infinite-dimensional lense space which we get as the orbit (quotient) space by the action of the group $\mathbf{Z}_m$ on the infinite-dimensional sphere. (Here the generator of the group assigns to a point $(z_1, z_2, \ldots) \in S^\infty$ the point

$$(e^{\frac{2\pi i}{m}} z_1, e^{\frac{2\pi i}{m}} z_2, \ldots),$$

where $|z_1|^2 + |z_2|^2 + \ldots = 1$ and $z_k = 0$ beginning from some k.)

Indeed, there exists a natural covering $p: S^\infty \to L_m$ with fibre $\mathbf{Z}_m$, therefore $\pi_1(L_m) = \mathbf{Z}_m$ and $\pi_i(L_m) = 0$ for $i > 1$.

Let us compute the integral homology of L_m. First of all we shall look for a cellular decomposition of L_m.

The sphere S^∞ has a special cellular decomposition with m cells in each dimension. In fact, denote by $\sigma_j^{2k} (j = 0, 1, \ldots, m-1)$ the set of all points $(z_1, z_2, \ldots)$ such that $z_{k+2} = z_{k+3} = \ldots = 0$, $z_{k+1} = \rho e^{i\varphi}$ where $\rho > 0$, $\varphi = \frac{2\pi j}{m}$; by σ_j^{2k+1} the set of all points $(z_1, z_2, \ldots)$ such that $z_{k+2} = z_{k+3} = \ldots = 0$ where $z_{k+1} = \rho e^{i\varphi}$, $\rho > 0, \frac{2\pi j}{m} < \varphi < \frac{2\pi(j+1)}{m}$. (The geometric interpretation is the following. The cell σ_0^{2k} is simply the upper half-sphere of the standardly imbedded sphere $S^{2k} \subset S^\infty$. Transformations from $\mathbf{Z}_m$ are rotations of this half-sphere, taking its base S^{2k-1} into itself. As a result we get m $2k$-dimensional half-spheres with a common base, which divide the sphere S^{2k+1} to m parts. These are the cells S_i^{2k} and S_i^{2k+1}, respectively.)

Clearly (provided the orientation of the cells is properly chosen) we have $\partial \sigma_j^{2k} = \sigma_0^{2k-1} + \ldots + \sigma_{m-1}^{2k-1}$ (the boundary of each cell σ_j^{2k} is their common base, which itself is divided to m cells) and $\partial \sigma_j^{2k+1} = \sigma_{j+1}^{2k} - \sigma_j^{2k}$.

The transformations from $\mathbf{Z}_m$ map the cells homeomorphically to each other. After the factorization process all cells $\sigma_j^{2k} (j = 0, \ldots, m-1)$ and $\sigma_j^{2k+1} (j = 0, 1, \ldots, m-1)$ are attached together. So the space L_m is divided into the cells $\sigma^0, \sigma^1, \sigma^2, \ldots$, one cell in each dimension, and

$$\partial \sigma^{2k} = m\sigma^{2k-1} \qquad (k = 1, 2, \ldots),$$

$$\partial \sigma^{2k-1} = 0 \qquad (k = 1, 2, \ldots).$$

It follows then that

$$H_i(L_m; \mathbf{Z}) = \begin{cases} \mathbf{Z}_m & \text{for } i = 2k+1 \\ 0 & \text{for } i = 2k \\ \mathbf{Z} & \text{for } i = 0 \end{cases}$$

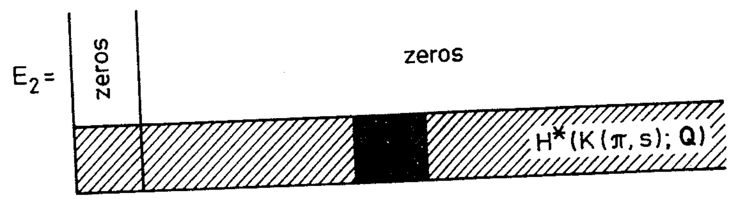
$$H^i(L_m; \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{for } i=0 \\ \mathbf{Z}_m & \text{for } i = 2k \\ 0 & \text{for } i = 2k+1 \end{cases}$$

i. e. $H^*(K(\mathbf{Z}_m, 1); \mathbf{Q}) = 0$.

Now we shall prove that $H^*(K(\pi, n); \mathbf{Q}) = 0\,(i>0, n>0)$ for any finite periodic group π. For $n=1$ the statement is proved; assume that it is valid for every $n \leq s-1$.

Consider the Serre fibration $* \sim E \xrightarrow{K(\pi,\, s-1)} K(\pi, s)$.

Here $s-1 \geq 1$, that is $s \geq 2$, therefore the base $K(\pi, s)$ is simply connected, and we may apply the Leray theorem. The fibre is cohomologically trivial by the assumption of induction (we are considering a spectral sequence over $\mathbf{Q}$) therefore all differentials are trivial, i. e. $E_2 = E_\infty$. On the other hand, $E_\infty = G(H^*(E; \mathbf{Q})) = 0$ i. e. $E_2 = 0$.



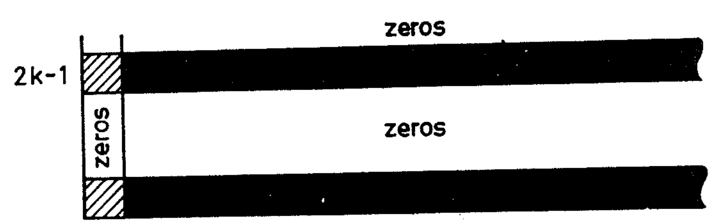
Thus $H^i(K(\pi, s); \mathbf{Q})) = 0$ for $i>0$, $s \geq 1$, what was to be proved. The first part of the theorem concerning cohomology of $K(\pi, n)$ with finite periodic groups, is proved.

Consider the Serre fibration $* \sim E \xrightarrow{K(\mathbf{Z},\, 2k-1)} K(\mathbf{Z}, 2k)$.

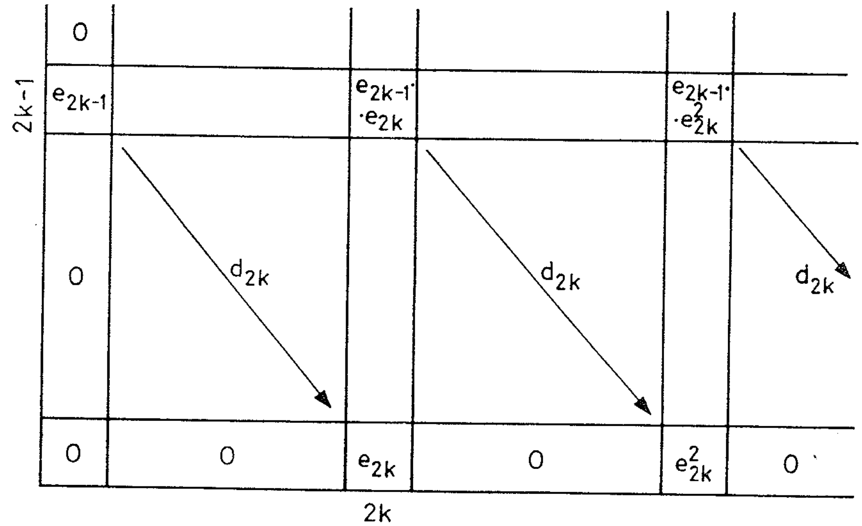
Let us now examine the second case of the theorem. The statement is proved for $K(\mathbf{Z}, 1)$. Assume that it is valid for all $s \leq 2k-1$ and, in particular:

$$H^i(K(\mathbf{Z}, 2k-1); \mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{for } \quad i=0,\, i=2k-1, \\ 0 & \text{for all other } i. \end{cases}$$

Write out the spectral sequence over $\mathbf{Q}$. The E_2 term looks as follows



We have $E_2 = E_3 = \ldots = E_{2k}$; $E_{2k+1} = E_{2k+2} = \ldots = E_\infty$. Then d_{2k} alone is different from zero.



As $E_{2k+1} = 0$, $d_{2k}\colon E_{2k}^{0,2k-1} \to E_{2k}^{2k,0}$ is an isomorphism. Therefore $E_{2k}^{2k,0} = \mathbf{Q}$ and $d_{2k}(e_{2k-1}) = e_{2k}$ where $e_{2k} \in E_{2k}^{2k,0}$ is the generator. We get $H^{2k}(K(\mathbf{Z}, 2k); \mathbf{Q}) = \mathbf{Q}$.

We already know the groups $E_2^{p,q}$ for $p \le 2k$: $E_2^{00} = E_2^{0,2k-1} = E_2^{2k,0} = E_2^{2k,2k-1} = \mathbf{Q}$, the others are trivial. The generator of the group $E_2^{2k,2k-1}$ is the product $e_{2k-1}e_{2k}$. From $E_{2k+1} = 0$ it follows that $E_{2k}^{q,0} = 0$ for $2k < q < 4k$ and $E_{2k}^{4k,0} = \mathbf{Q}$ while the generator of the latter is $d_{2k}(e_{2k-1}e_{2k}) = e_{2k}^2$. Hence $H^{4k}(K(\mathbf{Z}, 2k); \mathbf{Q}) = \mathbf{Q}$. Going on in the same line we get

$$H^i(K(\mathbf{Z}, 2k); \mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{for } i = 0, 2k, 4k, 6k, 8k, \dots, \\ 0 & \text{for all other } i \end{cases}$$

and the generator of the group $H^{2mk}(K(\mathbf{Z}, 2k); \mathbf{Q})$ is the element e_{2k}^m.

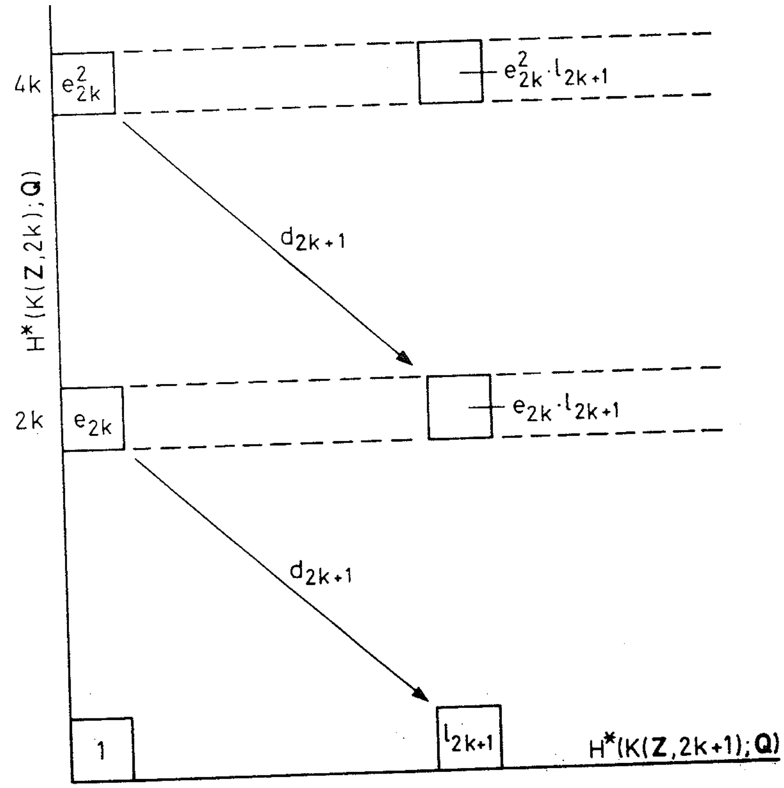
Finally consider the Serre fibration $* \sim E \xrightarrow{K(\mathbf{Z},\, 2k)} K(\mathbf{Z}, 2k+1)$.

The cohomology of $K(\mathbf{Z}, 2k)$ is already known. Let us look at E_2. (In the case of cohomology spectral sequences in the table for E_2 we sometimes write in the generators instead of the groups.)

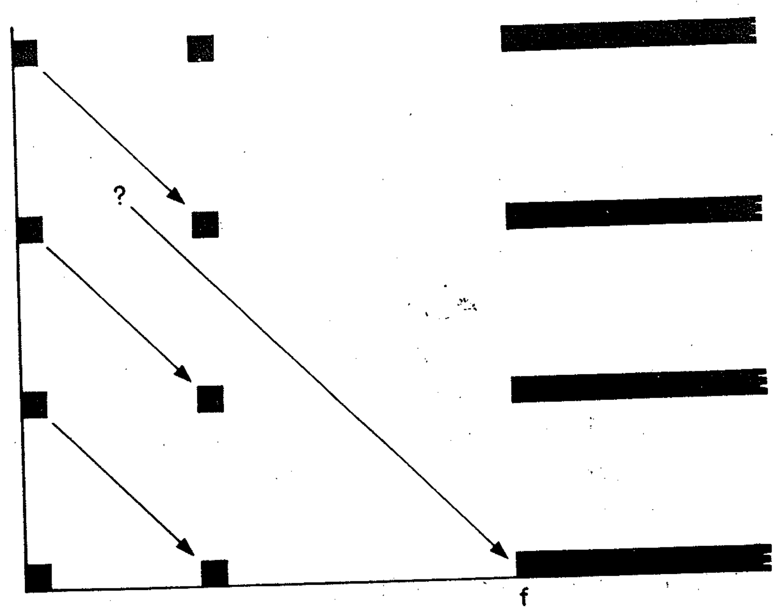
Since $E_\infty = 0$, we have $E_2^{p,0} = 0$ for $p < 2k+1$, $E_2^{2k+1,0} = E_{2k+1}^{2k+1,0} = \mathbf{Q}$ and so the differential $d_{2k+1}^{0,2k}\colon E_{2k+1}^{0,2k} \to E_{2k+1}^{2k+1,0}$ is an isomorphism. Therefore among the groups $E_2^{p,q} = E_{2k+1}^{p,q}$ with $q \le 2k+1$ the only ones different from zero are $E_2^{0,2mk}$ and $E_2^{2k+1,2mk}$ $(m = 0, 1, 2, \dots)$. Their generators are $e_{2k}^m \in E_2^{0,mk}$ and $e_{2k+1}e_{2k}^m \in E_2^{2k+1,2mk}$ where $e_{2k+1} = d_{2k+1}^{0,2k}(e_{2k})$. We have

$$d_{2k+1}^{0,2mk}(e_{2k}^m) = me_{2k}^{m-1} \cdot (d_{2k+1}^{0,2k} e_{2k}) = me_{2k}^{m-1} e_{2k+1}$$

i. e. $d_{2k+1}^{0,2mk}$ is an isomorphism. (It has been used here that the coefficients are from $\mathbf{Q}$ and so division by m is possible.)



Now in the bottom line we do not have a single nontrivial element to the right from the generator f. Indeed, suppose that there exists one and choose one having minimal dimension. Then between it and f there is a chain of zeros.



The element f must be in the image of one of the differentials. Now all elements to the left from it either are in the image of d_{2k+1} or are mapped by d_{2k+1} to some element certainly different from f. Then there is no nontrivial group to the right from e_{2k+1}.

Thus

$$H^i(K(\mathbf{Z}, 2k+1); \mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{for } i=0,\ 2k+1, \\ 0 & \text{for the other } i \end{cases}$$

Q.e.d.

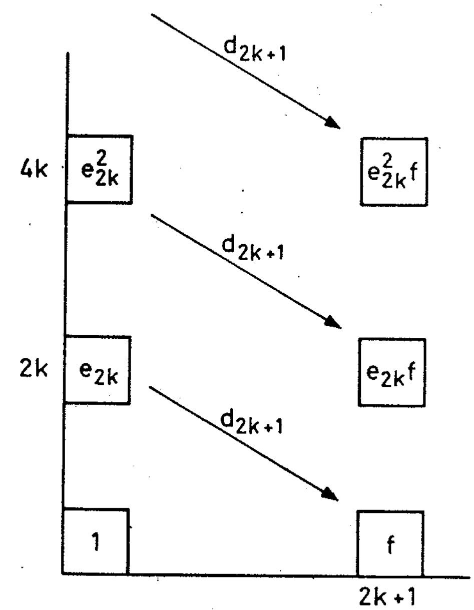
Let us show how to apply this result to find the ranks of the homotopy groups of spheres.

Consider the sphere S^{2k+1}. We have $\pi_{2k+1}(S^{2k+1})=\mathbf{Z}$. Examine the first killing space $X=S^{2k+1}|_{2k+1}$ of S^{2k+1}. (Thus $\pi_i(X)=\pi_i(S^{2k+1})$ for $i\geq 2k+2$ and $\pi_i(X)=0$ if $i<2k+2$).

Let us compute the rational cohomology of X. Consider the fibration

$$X = S^{2k+1}|_{2k+1} \xrightarrow{K(\mathbf{Z},\,2k)} S^{2k+1}.$$

Because the rational cohomology of S^{2k+1} and $K(\mathbf{Z}, 2k)$ are known we at once write the term E_2:



$d_2 = d_3 = \dots = d_{2k} = d_{2k+2} = \dots = 0$, i. e.

$$E_2 = E_3 = \dots = E_{2k+1}; \quad E_{2k+2} = E_{2k+3} = \dots = E_\infty.$$

Because $\pi_i(X) = 0$ for $i<2k+2$, we have $H^i(X; \mathbf{Q}) = 0$ for $i<2k+2$. This implies $E_\infty^{0,2k} = E_\infty^{2k+1,0} = 0$, that is, $d_{2k+1}^{0,2k}: E_{2k+1}^{0,2k} \to E_{2k+1}^{2k+1,0}$ is an isomorphism. Denote by f the generator $d_{2k+1}^{0,2k}(e_{2k}) \in E_{2k+1}^{2k+1,0}$. The generators of the groups $E_{2k+1}^{0,2mk}$ and $E_{2k+1}^{2k+1,2mk}$ $(m=0, 1, \dots)$ are e_{2k}^m and $e_{2k}^m f$ $(m=0, 1, \dots)$. We have $d_{2k+1}^{0,2mk}(e_{2k}^m) = m\cdot e_{2k}^{m-1} f$, i. e. every differential $d_{2k+1}^{0,2mk}$ is an isomorphism, and $E_\infty=0$. Thus $H^*(S^{2k+1}|_{2k+1}; \mathbf{Q})=0$, i. e. all integer cohomologies of $X=S^{2k+1}|_{2k+1}$ have finite order.

Now we give a lemma which will be important in our further investigations.

Lemma. Suppose that the killing space $Y|_q$ of a space Y has trivial rational cohomology. Then all subsequent space $Y|_t$, $t>q$ have trivial rational cohomology, too.

Indeed, consider the fibration

$$Y|_{q+1} \xrightarrow{K(\pi,\, q)} Y|_q$$

where $\pi=\pi_{q+1}(Y|_q) = H_{q+1}(Y|_q; \mathbf{Z})$ is a finite group by assumption. We have proved that $K(\pi, q)$ is cohomologically trivial over the rational.numbers whenever π is finite. Then the spectral sequence of the fibration is trivial, too. As $\mathbf{Q}$ is a field, we have $H^*(Y|_{q+1}; \mathbf{Q})=0$. Similarly $H^*(Y|_t; \mathbf{Q})=0$ for $t>q$. Q. e. d.

We note a consequence of the lemma. Assume that $H^*(Y|_q; \mathbf{Q}) = 0$; then $\pi_i(Y)$ is finite for all $i>q$.

This implies the following.

Theorem. All homotopy groups $\pi_i(S^{2k+1})$ of the odd-dimensional sphere S^{2k+1} are finite for $i>2k+1$.

This theorem made no use of the structure of the spheres i. e. we also have the following statement. Let X be a CW complex such that

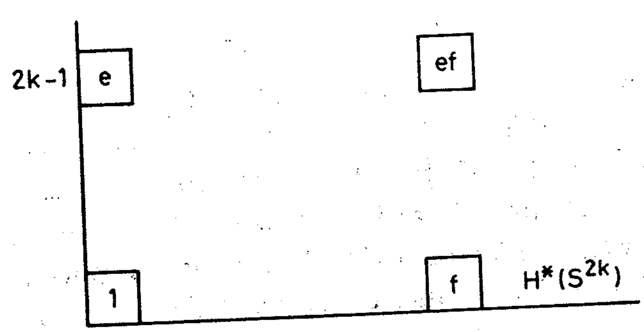
$$H^i(X; \mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{for } i = 0, 2k+1, \\ 0 & \text{for all other } i \end{cases}$$

and $\pi_1(X) = 0$. Then all homotopy groups $\pi_q(X)$ with $q \neq 2k+1$ are finite, and $\pi_{2k+1}(X)$ is a sum of $\mathbf{Z}$ and a finite group.

Indeed, $H^i(X|_{2k}; \mathbf{Q}) = H^i(X; \mathbf{Q})$, for $X|_{2k}$ has been obtained from X through a chain of fibrations whose cohomology over $\mathbf{Q}$ was trivial. Further, $\pi_{2k+1}(X|_{2k}) = H_{2k+1}(X|_{2k}; \mathbf{Z}) = \mathbf{Z} \oplus$ *finite group*.

Finally, $H^*(K(\pi_{2k+1}(X); 2k+1); \mathbf{Q}) \cong H^*(K(\mathbf{Z}, 2k+1); \mathbf{Q})$, and we can follow the same argument as above.

Let us now examine the case of even-dimensional spheres. Consider the fibration $S^{2k}|_{2k} \xrightarrow{K(\mathbf{Z},\, 2k-1)} S^{2k}$. For E_2 we have



Since $\pi_i(S^{2k}|_{2k}) = 0$ for $i<2k$, we have $E_\infty^{0,2k-1} = E_\infty^{2k,0} = 0$, and the differential $d_{2k}^{0,2k-1}$ is an isomorphism. Hence

$$H^i(S^{2k}|_{2k}; \mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{for } i = 0, 4k-1, \\ 0 & \text{for all other } i. \end{cases}$$

Then $S^{2k}|_{2k}$ is a space whose cohomology satisfies the above assumptions, i. e. $\pi_i(S^{2k}|_{2k})$ is finite for $i \neq 4k-1$ and $\pi_{4k-1}(S^{2k}|_{2k}) = \mathbf{Z} \oplus$ *finite group*.

Finally we conclude that

$$\pi_i(S^{2k}) = \begin{cases} \mathbf{Z} \oplus \text{ finite group for } i=4k-1 \\ \mathbf{Z} \quad \text{for } i = 2k \\ \text{finite group for } i \neq 2k,\ 4k-1 \end{cases}$$

We recall that in $\pi_{4k-1}(S^{2k})$ we found an element of infinite order. Now we see that the elements of this form (and the elements $[\mathrm{id}] \in \pi_n(S^n)$) are, up to proportionality, the only elements of infinite order in homotopy groups of spheres.

The theorem of H. Cartan and J. P. Serre

Assume that X is a simply-connected topological space such that $H^*(X; \mathbf{Q})$ is a free skew-commutative algebra (i. e. an algebra generated by a finite set of homogeneous elements $e_i \in H^{r_i}(X; \mathbf{Q})$, $i=1,2,\ldots,s$ with the relations of skew commutativity: $e_ie_j = (-1)^{r_ir_j}e_je_i$ for all i, j and with no other relations). The rank of the group $\pi_k(X)$ is equal to the number of the generators of degree k (i. e. the number of the $r_1, r_2, \ldots, r_s$ equal to k).

$H^*(X; \mathbf{Q})$ decomposes to a tensor product $\Lambda(x_1, x_2, \ldots, x_t) \otimes \mathbf{Q}[y_{t+1}, \ldots, y_s]$ where $\mathbf{Q}[y_{t+1}, \ldots, y_s]$ is the ring of polynomials of commuting generators of even degrees and Λ is the exterior algebra of generators of odd degrees. Let it be noted that if the Cartan–Serre theorem is applicable to a space X then either $H^*(X; \mathbf{Q})$ is an exterior algebra or X is infinite dimensional.

In quite a few particular cases we already know the theorem. For instance, if $H^*(X; \mathbf{Q}) = 0$ then the ranks of the groups $\pi_q(X)$ are equal to zero. The conditions of the theorem are also satisfied by the cohomology algebras of $K(\pi, n)$ for any π and n: it is a free skew-commutative algebra with rank π and generators of dimension n. For $n \geq 2$ we get rank $\pi_i(K(\pi, n)) = 0$ for $i \neq n$ and $=$ rank π for $i = n$ (as we already know).

The spaces $X = S^{2k+1}$ satisfy the conditions too. Thus the theorem implies the formula we proved above in this section:

$$\operatorname{rank} \pi_i(S^{2k+1}) = \begin{cases} 1 & \text{for } i = 2k+1, \\ 0 & \text{for } i \neq 2k+1. \end{cases}$$

On the other hand the theorem is not applicable to even-dimensional spheres S^{2k} because the square of the even-dimensional generator $e \in H^{2k}(S^{2k}; \mathbf{Q})$ is zero.

According to a theorem of Hopf (see Milnor J. W., Moore J.; On the Structure of Hopf algebra. Ann. of Math, 1965, v. 81, pp. 211–264) the cohomology algebra of any H-space (including any topological group) is a free skew-commutative algebra. We have shown, for instance, that $H^*(SU(n); \mathbf{Q}) = \Lambda(e_3, e_5, \dots e_{2n-1})$.

The Cartan–Serre theorem implies only that

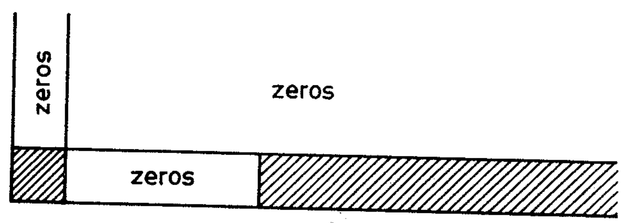
$$\pi_i(SU(n)) = \begin{cases} \mathbf{Z} \oplus \textit{ finite group} & \text{for } i = 3, 5, \dots, 2n-1, \\ \textit{finite group} & \text{for all other } i. \end{cases}$$

Proof the Cartan–Serre theorem

Suppose that $H^*(X; \mathbf{Q}) = \Lambda(e_1, e_2, \dots, e_t) \otimes \mathbf{Q}[e_{t+1}, \dots, e_s]$.

For the sake of definiteness we shall assume that the smallest among the degrees r_1, $r_2, \dots, r_s$ is an odd number $2k+1$. This means that in dimension $2k+1$, $H^*(X; \mathbf{Q})$ has some exterior multiplicative generators; let $e_1, e_2, \dots, e_m$ denote them. Thus $\deg e_i = 2k+1$ for $1 \le i \le m$ and $m \le t$.

By assumption, $\pi_1(X) = 0$. Since $H^2(X; \mathbf{Q}) = 0$, $H_2(X; \mathbf{Z}) = \pi_2(X)$ is finite. Consider the fibration $X|_2 \xrightarrow{K(\pi_2(X), 1)} X$. Its fibre has trivial cohomology over $\mathbf{Q}$. Then the E_2 term of its cohomology spectral sequence with coefficients in $\mathbf{Q}$ is of the form



Hence $H^*(X|_2; \mathbf{Q}) = H^*(X; \mathbf{Q})$. Similarly we get $H^*(X|_{2k}; \mathbf{Q}) = H^*(X; \mathbf{Q})$. The space $X|_{2k}$ is $2k$-connected by definition, and $\pi_i(X|_{2k}) = \pi_i(X)$ for $i \ge 2k+1$.

In other words, the ranks of the homotopy groups of X and $X|_{2k}$ coincide in all dimensions. Let us calculate $\pi_{2k+1}(X)$. We have $\pi_{2k+1}(X) = \pi_{2k+1}(X|_{2k}) = H_{2k+1}(X|_{2k}; \mathbf{Z})$. On the other hand, $H^{2k+1}(X; \mathbf{Q}) = H^{2k+1}(X|_{2k}; \mathbf{Q}) = \oplus_1^m \mathbf{Q}$. Then $H^{2k+1}(X|_{2k}; \mathbf{Z}) = \oplus_1^m \mathbf{Z}$ (it is torsion-free, for $\pi_{2k}(X|_{2k}) = 0$), and $H_{2k+1}(X|_{2k}; \mathbf{Z}) = (\oplus_1^m \mathbf{Z}) \oplus$ *finite group* (the finite group may come from $H^{2k+2}(X|_{2k}; \mathbf{Z})$). By the Hurewicz theorem, $\pi_{2k+1}(X|_{2k}) = (\oplus_1^m \mathbf{Z}) \oplus$ *finite group*, i. e. rank $\pi_{2k+1}(X) = n$ (equal to the number of multiplicative generators of degree $2k+1$ in $H^*(X; \mathbf{Q})$).

Now we are going to get rid of the free generators of dimension $2k+1$. We shall construct a chain of killing spaces $X|_{2k+1}^1$, $X|_{2k+1}^2$, $\dots$, $X|_{2k+1}^m$ wiping out one generator in each step.

As we know $\pi_{2k+1}(X|_{2k}) = \oplus_1^m \mathbf{Z} \oplus$ *finite group*. To each generator $e_1, e_2, \ldots, e_m$ of $H^{2k+1}(X|_{2k}; \mathbf{Q})$ there corresponds a generator $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_m$ of $\pi_{2k+1}(X|_{2k})$. Take $\tilde{e}_1$ and construct an imbedding of $X|_{2k}$ in $K(\mathbf{Z}, 2k+1)$ with respect to it. We shall not only glue up all homotopy groups of $X|_{2k}$ beginning from the dimension $2k+2$ but also all the generators in $\pi_{2k+1}(X|_{2k})$ except $\tilde{e}_1$. Then the homomorphism of homotopy groups induced by the imbedding $i\colon X|_{2k} \to K(\mathbf{Z}, 2k+1)$ maps all generators except $\tilde{e}_1$ into zero, while $\tilde{e}_1$ will be mapped on the generator of the group $\pi_{2k+1}(K(\mathbf{Z}, 2k+1))$.

Let us examine the induced fibration:

$$\begin{array}{ccc} X|^1_{2k+1} & \longrightarrow & E \sim * \\ {\scriptstyle K(\mathbf{Z},\,2k)}\downarrow & & \downarrow{\scriptstyle K(\mathbf{Z},\,2k)} \\ X|_{2k} & \xrightarrow{X|^1_{2k+1}} & K(\mathbf{Z}, 2k+1) \end{array}$$

We show that $\pi_i(X|^1_{2k+1}) = \pi_i(X|_{2k})$ for $i \geq 2k+2$ and $\pi_{2k+1}(X|^1_{2k+1}) = \pi_{2k+1}(X|_{2k})/(\tilde{e}_1)$ where $(\tilde{e}_1)$ is the subgroup generated by $\tilde{e}_1$; i. e. $\pi_{2k+1}(X|^1_{2k+1}) = \oplus_1^{m-1} \mathbf{Z} \oplus$ *finite group*.

Indeed, consider the segment

$$0 \to \pi_{2k+1}(X|^1_{2k+1}) \to \pi_{2k+1}(X|_{2k}) \xrightarrow{\alpha} \pi_{2k+1}(K(\mathbf{Z}, 2k+1) \to \pi_{2k}(X|_{2k+1}) \to 0$$

of the exact homotopy sequence of the fibration $X|_{2k} \xrightarrow{X|^1_{2k+1}} K(\mathbf{Z}, 2k+1)$.

By construction, $\alpha(\tilde{e}_1) = e$ (the generator of $\pi_{2k+1}(K(\mathbf{Z}, 2k+1))$) and $\alpha(\tilde{e}_i) = 0$ for $2 \leq i \leq m$. Then $\pi_{2k}(X|^1_{2k+1}) = 0$ and $\pi_{2k+1}(X|^1_{2k+1}) \cong \operatorname{Ker} \alpha$ where $\operatorname{Ker} \alpha$ is the group spanned on all generators of $\pi_{2k+1}(X|_{2k})$ except one, namely $\tilde{e}_1$.

So the group $\pi_{2k+1}(X|^1_{2k+1})$ has the free generators $\tilde{e}_2, \ldots, \tilde{e}_m$. Again we construct a similar space $X|^2_{2k+1}$ such that $\pi_i(X|^2_{2k+1}) = \pi_i(X|^1_{2k+1})$ for $i \neq 2k+1$ and $\pi_{2k+1}(X|^2_{2k+1}) = \pi_{2k+1}(X|^1_{2k+1})/(\tilde{e}_2)$. Thereafter we go on constructing the spaces $X|^3_{2k+1}$, X^4_{2k+1}, etc.

At some step of this processing we shall find that some term $X|^m_{2k+1}$ of the chain $X|^1_{2k+1}, X|^2_{2k+1}, \ldots,$ has finite π_{2k+1}. We kill this finite group too. The resulting space $X|_{2k+1}$ has the same rational cohomology as $X|^m_{2k+1}$. Its homotopy groups are

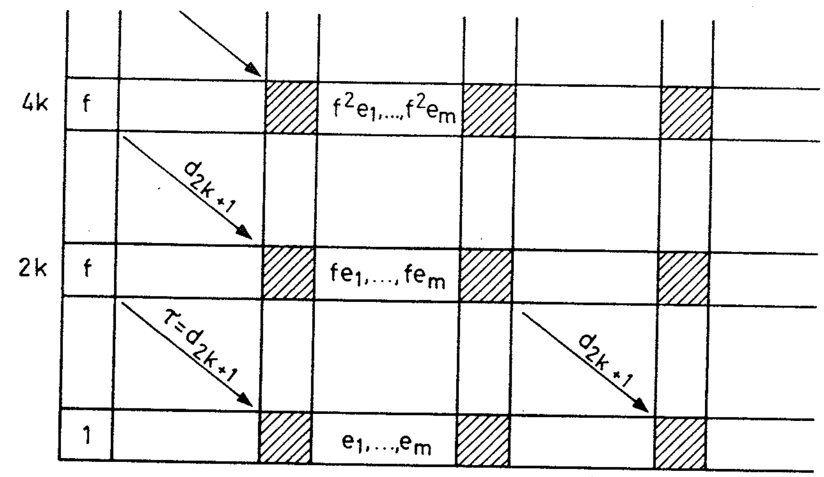
$$\pi_i(X|_{2k+1}) = \begin{cases} \pi_i(X) & \text{for } i > 2k+1 \\ 0 & \text{for } i \leq 2k+1. \end{cases}$$

(This space is homotopy equivalent to $X|_{2k+1}$ defined in the previous Section. We do not prove this as we are not going to use it. However, we use the same notation for the new space. The reason why we prefer the latter construction is the following. We are trying to find the rational cohomology of $X|_{2k+1}$ and to reveal that we are again within the conditions of the Cartan–Serre theorem. The familiar fibration $X|_{2k+1} \to X|_{2k}$ is inconvenient for this purpose as it has a fibre too huge, which makes the computations difficult to manage. The alternative way we choose allows us to exhaust this hugeness by small portions.)

Let us compute the rational cohomology of $X|_{2k+1}$ i. e. of $X|^m_{2k+1}$. Consider the fibration

$$X|^1_{2k+1} \xrightarrow{K(\mathbf{Z}, 2k)} X|_{2k}.$$

For E_2 we have

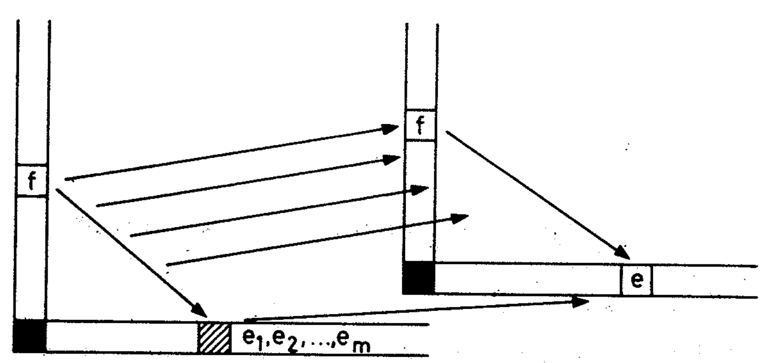


In column zero we have $H^*(K(\mathbf{Z}, 2k); \mathbf{Q}) = \mathbf{Q}[f]$ where $\deg f = 2k$. Obviously $E_2 = \ldots = E_{2k+1}$.

The transgression τ sends f into e_1. To prove this, consider the homomorphism of spectral sequences induced by the mapping

$$\begin{array}{ccc} X|^1_{2k+1} & \longrightarrow & E \\ K(\mathbf{Z}, 2k) \downarrow & \curvearrowright & \downarrow K(\mathbf{Z}, 2k) \\ X|_{2k} & \longrightarrow & K(\mathbf{Z}, 2k+1) \end{array}$$

We obtain



The generator $e \in H^{2k+1}(K(\mathbf{Z}, 2k+1); \mathbf{Q})$ is mapped into $e_1 \in H^{2k+1}(X|_{2k}; \mathbf{Q})$ and f into f. Therefore $\tau(f) = e_1$ in the spectral sequence.

We conclude that neither column zero nor the part of column $(2k+1)$ above e_1 will pass into the E_{2k+2} term (and even less into E_∞), for we have $d_{2k+1}(f^q) = qf^{q-1}e_2$, i. e. $f^q e_1 = d(\frac{1}{q+1} f^{q+1})$.

Consider the ideal $J(e_1)$ in $H^*(X|_{2k}; \mathbf{Q})$ generated by the element e_1, and the subalgebra H which consists of all elements not containing e_1. Since the algebra

$$H^*(X|_{2k}; \mathbf{Q})$$

is free, it is the direct sum of the modules $H \oplus J(e_1)$. As e_1 is an exterior generator, we have $J(e_1) = e_1 H$. Multiplication in $J(e_1)$ is trivial.

The intersection of $\operatorname{Im} d_{2k+1}$ with the bottom row of E_{2k+1} coincides with the ideal $J(e_1)$. Indeed, let $x \in J(e_1)$, i. e. $x = e_1 P$. Then $x = (df)P = d(fP)$ (as $dP = 0$).

Let $h \in H^*(X|_{2k}; \mathbf{Q})$ and $h \in \operatorname{Im} d_{2k+1}$, i. e. $h = d_{2k+1}(\omega)$. On the other hand, $\omega = f\rho$ where $\rho \in H^*(X|_{2k}; \mathbf{Q})$, i. e. $h = (df)\rho = e_1\rho \in J(e_1)$. In other words when we pass from E_{2k+1} to E_{2k+2} we obtain in the first row the algebra $H = H^*(X|_{2k}; \mathbf{Q})/J(e_1)$.

What stands in the upper rows of E_{2k+2}? Each element of $E^{p,q}_{2k+1}$ for $q > 0$ is of the form $f^s(x+y)$ where $x \in J(e_1)$ and $y \in H$, i. e. $x = e_1 x'$. Now $f^s x = d_{2k+1}(\frac{1}{s+1} f^{s+1} x)$ and $d_{2k+1}(f^s y) = sf^{s-1}xy \neq 0$. Thus there remains nothing in $E^{p,q}_{2k+2}$ with $q > 0$, i. e. the only nontrivial groups contained in E_{2k+2} are in the bottom row, and $\oplus_p E^{p,0}_{2k+2} = H = H^*(X|_{2k}; \mathbf{Q})/J(e_1)$. By consideration of dimensions we obtain $E_{2k+2} = E_\infty = H^*(X|^1_{2k+1}; \mathbf{Q})$. Hence

$$H^*(X|^1_{2k+1}; \mathbf{Q}) \cong H^*(X; \mathbf{Q})/J(e_1) = \wedge(e_2, e_3, \ldots, e_t) \otimes \mathbf{Q}[e_{t+1}, \ldots, e_s].$$

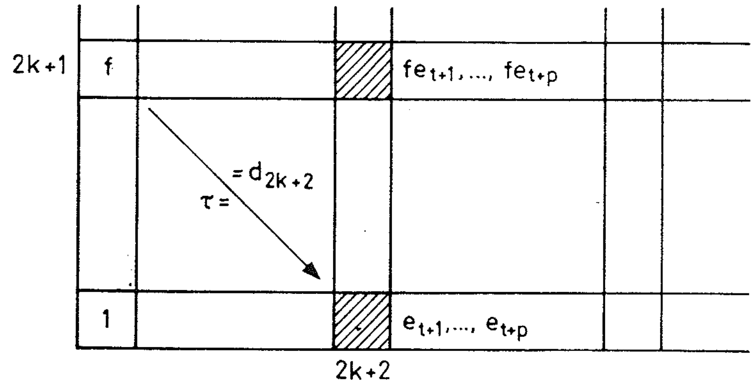
We see that by constructing the space $X|^1_{2k+1}$ we have killed together with a generator $\pi_{2k+1}(X)$, a multiplicative generator of $H^*(X; \mathbf{Q})$. By repeating this construction we get a space $X|^m_{2k+1}$ for which $H^*(X|^m_{2k+1}; \mathbf{Q}) = H^*(X|_{2k+1}; \mathbf{Q}) = \wedge(e_{m+1}, \ldots, e_t) \otimes \mathbf{Q}[e_{t+1}, \ldots, e_s]$, i. e. we remain within the condition of the Cartan–Serre theorem. This time however there are no generators in dimension $2k+1$.

Let H^{2k+2} contain the even-dimensional generators $e_{t+1}, e_{t+2}, \ldots, e_{t+p}$ where $t + p \leq s$. We get rid of them step-by-step by constructing the killing spaces $X|^1_{2k+2}, \ldots, X|^p_{2k+2}$. By literally repeating the above construction we obtain a spectral sequence such that

$$E_2 = H^*(X|_{2k+1}; \mathbf{Q}) \otimes H^*(K(\mathbf{Z}, 2k+1); \mathbf{Q}) = H^*(X|_{2k+1}; \mathbf{Q}) \otimes \wedge(f).$$

where $\deg f = 2k+1$. Again it turns out that $d_{2k+2}(f) = e_{t+1}$ (assuming that $X|^1_{2k+2}$ kills the generator $\pi_{2k+2}(X)$ which corresponds to e_{t+1})

Once again we have the direct sum (of modules) $H^*(X|_{2k+1}; \mathbf{Q}) = H \oplus J(e_{t+1})$ where H consists of the elements not containing e_{t+1}. Let it be noted that, unlike in the



previous case, multiplication in $J(e_{t+1})$ is not trivial and $J(e_{t+1}) \neq e_{t+1}H$. As before, the bottom row alone remains in E_∞. It consists of

$$H^*(X|^1_{2k+2}; \mathbf{Q}) = H^*(X|_{2k+1}; \mathbf{Q})/J(e_{t+1}) =$$
$$= \wedge(e_{m+1}, e_{m+2}, \ldots, e_t) \otimes \mathbf{Q}[e_{t+2}, e_{t+3}, \ldots, e_{t+p}, \ldots, e_s],$$

i. e. a further multiplicative generator has disappeared.

By carrying out this argument successively we show that

$$H^*(X|_{2k+2}; \mathbf{Q}) = \wedge(e_{m+1}, \ldots, e_t) \otimes \mathbf{Q}[e_{t+p+1}, \ldots e_s].$$

Further we consider $X|_{2k+3}$, $X|_{2k+4}$, ..., etc. and obtain the theorem. Q.e.d.

Some remarks concerning the Cartan–Serre theorem

(1) The theorem does not apply to every space. As we observed on the example of even-dimensional sphere the method of proving the theorem may be universally applied for computing the ranks of homotopy groups. To get the exact answer for any simply-connected space, i. e. to express the ranks of the homotopy groups in terms of the cohomology algebra, is however far from easy.* Nevertheless we have for every simply-connected space the formula

$$\pi_i(X) \otimes \mathbf{Q} = H^i(X; \mathbf{Q})$$

for $i < 2n-1$ if $\pi_2(X) = \ldots = \pi_{n-1}(X) = 0$ (and also if $\pi_2(X), \ldots, \pi_{n-1}(X)$ are finite). The proof is similar to, and even easier than, the Cartan–Serre theorem, as one has to examine such dimensions where the multiplicative structure of $H^*(X; \mathbf{Q})$ has no effect.

We mention an important consequence of this theorem. By the generalized Freudenthal theorem (see §20) the group $\pi_{N+i}(\Sigma^N X)$ with $N > i+1$ does not depend on

* An adequate theory of rational homotopy types (which may be regarded as proper generalization of the Cartan–Serre theorem) was developed in the late 70's by D. Sullivan.

N. Let it be denoted by $\pi_i^S(X)$ and called the i-th stable homotopy group of X. By the above,

$$\pi_i^S(X) \otimes \mathbf{Q} = H^i(X; \mathbf{Q})$$

for every i. Indeed, $\pi_i^S(X) \otimes \mathbf{Q} = \pi_{N+i}(\Sigma^N X) \otimes \mathbf{Q} = H^i(X; \mathbf{Q})$ if $N > i+1$, i. e. $N+i < 2N-1$, since $\pi_i(\Sigma^N X) = 0$, for $i < N$.

(2) In the statement of the Cartan–Serre theorem we assumed that $\pi_1(X) = 0$. In the course of the proof we noted the point where this assumption was exploited. Actually the theorem is valid in much more general circumstances, namely, it is sufficient to require (beside the condition on the structure of the cohomology) simplicity of X, i. e. that $\pi_1(X)$ has trivial action on the groups $\pi_r(X)$, $r \geq 1$.

Such are, for example, all H-spaces, including all Lie groups.

The condition that X is simple is essential, as there are many examples of spaces with "good" rational cohomology for which the Cartan–Serre theorem is not valid. Indeed, let $X = \mathbf{RP}^2$. Then $\pi_1(X) = \mathbf{Z}_2$ and X is not simple. (The generator $\alpha \in \pi_1(X)$ acts on $\pi_2(X)$ as multiplication by -1.) The rational cohomology groups are trivial in the positive dimensions, and so, had it been applied, the theorem would say that all homotopy groups of X are finite. Actually $\pi_2(\mathbf{RP}^2) = \pi_2(S^2) = \mathbf{Z}$. Interestingly the effect on the rational cohomology is made by a finite fundamental group.

Let us sketch the proof of the Cartan–Serre theorem under the assumption that X is simple, without going into details.

Assume that X is simple and $\pi_1(X) = G$. Simplicity of X implies that G is commutative (commutativity of the fundamental group is equivalent to 1-simplicity). Now $H_1(X; \mathbf{Z})$ is the abelianization of the fundamental group, $H_1(X; \mathbf{Z}) = G$ and rank $\pi_1 =$ rank $H_1 =$ rank H^1 which is equal to the number of one-dimensional generators of $H^*(X; \mathbf{Q})$; in other words, the theorem is valid for $\pi_1(X)$. Let the generators of $H^1(X; \mathbf{Q})$ be denoted by $e_1, e_2, \ldots, e_k$. Consider the universal covering $p\colon T \xrightarrow{G} X$. We shall prove the following

Lemma.

$$H^*(T; \mathbf{Q}) = H^*(X; \mathbf{Q})/(e_1, e_2, \ldots, e_k)$$

where $(e_1, e_2, \ldots, e_k)$ is the ideal generated by the one-dimensional generators. (If we had only "unfolded" a single generator e_1 rather than constructed the universal covering, we should have $H^*(T_{e_1}; \mathbf{Q}) = H^*(X; \mathbf{Q})/(e_1)$.)

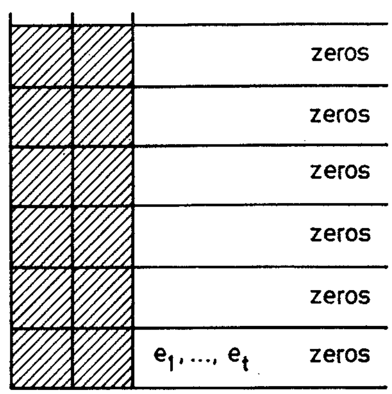
The theorem immediately follows from the lemma, as $\pi_1(T) = 0$, $\pi_k(T) = \pi_k(X)$ for $k \geq 2$, $H^*(T; \mathbf{Q})$ is free skew-symmetric and so the "simply-connected" Cartan–Serre theorem can be applied.

Consider the imbedding $X \subset K(G, 1)$ induced by the isomorphism of fundamental groups. Consider the homotopy equivalent fibration $X \to K(G,1)$. The fibre is homotopy equivalent to T. (This is the analogue of the equivalence of the two variants of killing spaces: as spaces of fibration with fibres $K(\pi_n(X), n-1)$ or as fibres of fibrations with bases $K(\pi_n(X), n)$.) Now we have the fibration $X \xrightarrow{T} K(G, 1)$. We cannot however immediately claim that E_2 is $H^*(K(G, 1); H^*(T))$ since this statement has only been proved when the base space was simply connected. Let it be recalled, though, that the important role was played not by the base but by the following property of the fibre: any paths connecting a pair of points x, y in the base induce homotopic mappings $F_x \to F_y$. Or alternatively: Any closed path in the base with the beginning and the end in the point x induces a mapping $F_x \to F_x$ homotopic to the identity. In this case it is ensured by the simplicity of X: the action of the fundamental group $K(G, 1)$ on the fibre T of the fibration coincides (up to homotopy) with that of $\pi_1(X) = G$ on T as of monodromy group; the latter defines on $\pi_r(T)$ the same automorphisms as $\pi_1(X)$ does on $\pi_r(X) = \pi_r(T)$, i. e. the identity automorphisms.

Thus $E_2 = H^*(K(G, 1); H^*(T))$. We have

$$H^q(K(G, 1); \mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{for } q=0 \\ \underbrace{\mathbf{Q} \oplus \ldots \oplus \mathbf{Q}}_{\operatorname{rank} G} & \text{for } q=1 \\ 0 & \text{for } q>1 \end{cases}$$

The term E_2 has the form



Obviously $E_2 = E_\infty$ (by dimensional consideration) and $H^*(X) = H^*(T) \otimes \Lambda(e_1, e_2, \ldots, e_k)$. Q.e.d.

(3) The condition of simplicity of X may still be weakened. Actually it is sufficient to demand that for any $\alpha \in \pi_1(X)$ and $\beta \in \pi_r(X)$ the difference $\alpha(\beta) - \beta$ is an element of finite order in $\pi_r(X)$. This already ensures that any transformation from the monodromy group induces the identity mapping of $H^*(T; \mathbf{Q})$.

§25. THE RING $H^*(K(\pi, n); \mathbf{Z}_p)$

Thus far we only made use of the information about the rational cohomology of $K(\pi, n)$. Now we shall need $H^*(K(\pi, n); \mathbf{Z}_p)$.

As we have seen the sequence of killing spaces

$$X \leftarrow X|_n \leftarrow X|_{n+1} \leftarrow \dots$$

combined with our informations about $H^*(K(\pi, n); \mathbf{Q})$ enable us to determine the free components of the homotopy groups of X. The cohomology of $K(\pi, n)$ mod p will be needed to find the torsion of the homotopy groups. (The integral cohomology of $K(\pi, n)$ is known, too, but we are not going to study them.)

Computing $H^*(K(\mathbf{Z}_p, n); \mathbf{Z}_p)$

We already have the topological description of the complex $K(\mathbf{Z}_p; 1)$, namely $K(\mathbf{Z}_p, 1) = L_p$. We know the cell structure of it and without difficulty can describe the cohomology structure.

Theorem. For arbitrary prime numbers p and p'

$$H^i(K(\mathbf{Z}_p, 1); \mathbf{Z}_{p'}) = \begin{cases} 0 & \text{if } p \neq p' \text{ and } i > 0, \\ \mathbf{Z}_p & \text{if } p = p' \text{ and for all } i \geq 0. \end{cases}$$

If $p = p'$, multiplication in $H^*(K(\mathbf{Z}_p, 1); \mathbf{Z}_p)$ is the following. There exist in $H^i(K(\mathbf{Z}_p, 1); \mathbf{Z}_p)$ $(i = 1, 2, \dots)$ generators e_i $(i = 1, 2, \dots)$ such that

(1) for $p \neq 2$, $e_3 = e_1 e_2$, $e_4 = e_2^2$, $e_5 = e_1 e_2^2$, $e_6 = e_2^3, \dots, e_1^2 = 0$ i. e.

$$H^*(K(\mathbf{Z}_p; 1); \mathbf{Z}_p) = \mathbf{Z}_p[e_2] \otimes \Lambda(e_1).$$

(2) for $p = 2$, $e_i = e_1^i$ for all i, i. e.

$$H^*(K(\mathbf{Z}_2, 1); \mathbf{Z}_2) = \mathbf{Z}_2[e_1].$$

It will be recalled that $K(\mathbf{Z}_p, 1)$ may be decomposed into the cells $\sigma^0, \sigma^1, \dots,$ one cell in each dimension, such that

$$[\sigma^{i+1} : \sigma^i] = \begin{cases} 0 & \text{if } i \text{ is even}, \\ p & \text{if } i \text{ is odd}. \end{cases}$$

The "additive" part of the theorem immediately follows from this. The "multiplicative" part is proved thus far for $p = 2$ $(K(\mathbf{Z}_2, 1) = \mathbf{RP}^\infty)$.

Let $p > 2$. Consider the fibration $\pi\colon L_p \to \mathbf{CP}^\infty$ (the mapping π assigns to the point

$$(z_0, z_1, \dots) = (z_0 e^{\frac{2\pi i}{p}}, z_1 e^{\frac{2\pi i}{p}}, \dots) = \dots = (z_0 e^{\frac{2\pi i}{p}(p-1)}, z_1 e^{\frac{2\pi i}{p}(p-1)}, \dots)$$

the point $(z_0 : z_1 : \dots) \in \mathbf{CP}^\infty$) with the fibre S^1. The E_2 term in the spectral sequence is

f	0	fe_2	0	fe_2^2	0	fe_2^3	0	...
1	0	e_2	0	e_2^2	0	e_2^3	0	...

In view of our knowledge the groups $H^q(L_p; \mathbf{Z}_p)$ we conclude that $E_2 = E_3 = \ldots = E_\infty$ and in $H^q(L_p; \mathbf{Z}_p) = \mathbf{Z}_p$ generators e_q can be selected such that $e_{2k} = e_2^k$ and $e_1 e_{2k} = e_{2k+1}$. Moreover $e_1^2 = 0$, since $e_1^2 = -e_1^2$ (because of the skew symmetry of multiplication in the cohomology). Q. e. d.

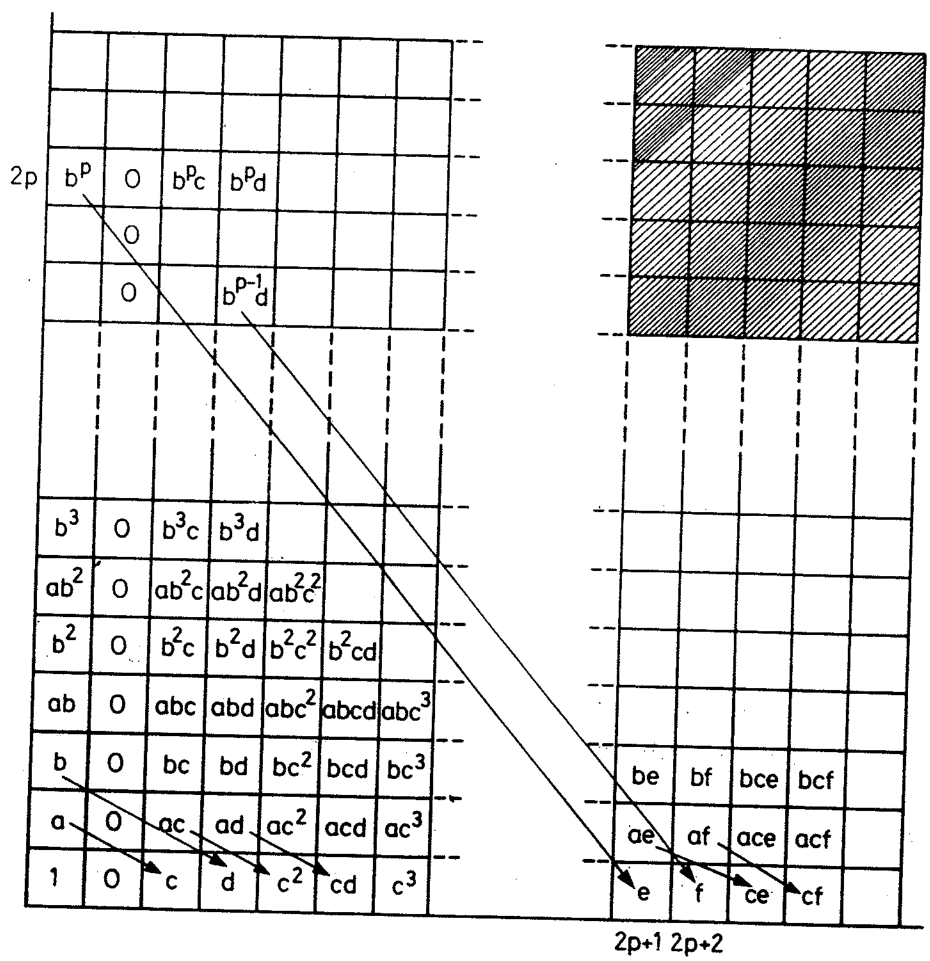
Now we compute the ring $H^*(K(\mathbf{Z}_p, n); \mathbf{Z}_p)$ where p and p' are prime numbers. Assume that $p \neq p'$. The cohomology groups of $K(\mathbf{Z}_p, 1)$ mod p' are trivial. Suppose the same is true for $K(\mathbf{Z}_p, n-1)$. Consider the fibration

$$* \sim E \xrightarrow{K(\mathbf{Z}_p, n-1)} K(\mathbf{Z}_p, n)$$

The total space is contractible, hence $K(\mathbf{Z}_p, n)$ is cohomology trivial mod p', too. Then $H^q(K(\mathbf{Z}_p, n); \mathbf{Z}_{p'}) = 0$ for $q > 0$ provided $p \neq p'$. Consider $H^*(K(\mathbf{Z}_p, n); \mathbf{Z}_p)$. At first we study the spectral sequence of cohomology mod p of the fibration

$$* \sim E \xrightarrow{K(\mathbf{Z}_p, 1)} K(\mathbf{Z}_p, 2).$$

For E_2 we have



Since $\pi_1(K(\mathbf{Z}_p, 2)) = 0$, we have $E_2^{1,q} = 0$ for every q. In the column zero there is

$$H^*(K(\mathbf{Z}_p, 1); \mathbf{Z}_p) = \begin{cases} \mathbf{Z}_p[b] \otimes \wedge(a) & \text{if } p \neq 2 \\ \mathbf{Z}_p[a] & \text{if } p = 2; \end{cases}$$

here deg $a = 1$, deg $b = 2$. As the total space of the fibration is contractible, $E_\infty = 0$, and so $d_2(a) \neq 0$. Let $d_2 a$ be denoted by c. Then c is obviously the only generator of $E_2^{2,0}$. Thus the second column in E_2 is obtained by multiplying the column zero by c. What is the image of b?

As $d_2(b)$ belongs to the group $E_2^{2,1}$ which has the generator ac, we have $d_2(b) = k \cdot ac$ where $k \in \mathbf{Z}_p$. We prove that $k = 0$. Consider again $d_2 \colon 0 = d_2^2 b = k d_2 ac = kc^2$.

If $k \neq 0$ in $\mathbf{Z}_p$ then $c^2 = 0$ in $H^4(K(\mathbf{Z}_p, 2); \mathbf{Z}_p)$. This will immediately lead to contradiction. The element $c \in H^2(K(\mathbf{Z}_p, 2); \mathbf{Z}_p)$ is known to have the property that for any space X and any $\alpha \in H^2(X; \mathbf{Z}_p)$ there exists a mapping $f \colon X \to K(\mathbf{Z}_p, 2)$ such that $f^*(c) = \alpha$ (see §17, p. 117; we used the notation e instead of c there). Then $f^*(c^2) = \alpha^2$ and we obtain for any X and $\alpha \in H^2(X; \mathbf{Z}_p)$ the relation $\alpha^2 = 0$ which cannot be true, as seen on the example of $\mathbf{CP}^\infty$. Thus we have proved that $c^2 \neq 0$ in E_2. Literally the same argument may be used to show $c^m \neq 0$ for any integer m. (This will be important in the sequel.)

We have obtained that $d_2(b) = 0$, i. e. b is mapped into E_3.

Now $E_\infty = 0$, therefore by consideration of dimension $d_3 b \neq 0$. Let $d_3 b$ be denoted by d. There are no generators in $E_2^{2,0}$ and $E_3^{3,0}$ but c and d; by the same reason $E_\infty = 0$.

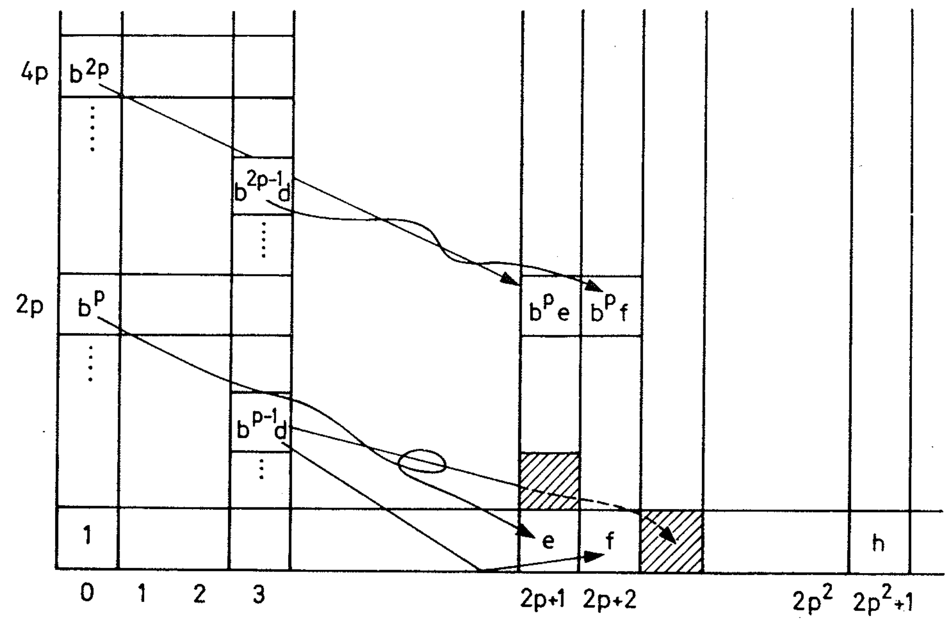
The differential $d_2^{1,1} \colon E_2^{1,1} \to E_2^{3,0}$ is trivial ($E_2^{1,1} = 0$) hence $E_2^{3,0} = E_3^{3,0}$, i. e. the generator d comes into $E_3^{3,0}$, from the isomorphic group $E_2^{3,0}$, i. e. we have shown that $H^3(K(\mathbf{Z}_p, 2); \mathbf{Z}_p) = \mathbf{Z}_p$.

As we see two generators c and d appear in the zero row of E_2. They stand next to each other and have degrees 2 and 3. Assume that $p \neq 2$, then $d^2 = 0$ (d is of odd dimension).

We already know that $c^m \neq 0$ for any m. We prove that $c^m d \neq 0$ for $m < p$ in E_2. Indeed, let $c^m d = 0$ for some $m < p$, while $c^l d \neq 0$ for $l < m$. The group $\bigoplus_{\substack{0 \leq p \leq 2m+2 \\ 0 \leq q < \infty}} E_2^{p,q}$ is additively generated by the elements of type $s^{\varepsilon_1} b^s c^t d^{\varepsilon_2}$ where $\varepsilon_1 = 0$ or 1 $0 \leqslant p \leqslant 2m+2$, $\varepsilon_2 = 0$ or 1 s is arbitrary, $t \leqslant m+1$ for $\varepsilon_2 = 0$ and $t \leqslant m-1$ for $\varepsilon_2 = 1$, and we have also $d_2(ab^s c^t d^{\varepsilon_2}) = b^s c^{t+1} d^{\varepsilon_2}$, $d_3(b^s) = s b^{s-1} d$. (If there were, in addition to c and d, a further multiplicative generator in the bottom row in a dimension $\leqslant 2m+2$, it would also remain in E_∞, since no element standing to the left could be carried into it.) The element $ac^{m-1}d$ cannot be therefore the image of any differential. (Those elements which might be sent into $ac^{m-1}d$, as dimensional considerations permit, according to the formulas above, either go into some other elements or themselves are images of differentials.) The only possibility that is left for $ac^{m-1}d$ not remaining in E_∞ is that $d_2(ac^{m-1}d) \neq 0$. Now $d_2(ac^{m-1}d) = c^m d$, hence $c^m d \neq 0$.

In the first $2p$ column thus almost all elements are killed by the second and third differentials. What remains in E_4?

Clearly the only elements that remain are b^{kp} (for all k) and $b^{kp-1}d$ (as $d_3 b^k p = kpb^{kp-1}d = 0$). They may not be sent by the differentials into the first $2p$ columns since all elements are "occupied" there: those of the form $ab^s c^t d^r$ are not cycles with respect to the second differential; those of the form $b^s c^t d^r$ for $t \geq 0$ are in the image of the second differential; the elements b^s and $b^s d$, with the exception of b^{kp} and $b^{kp-1}d$ are killed by the third differential. Here is the diagram of E_4:



Here b^p (b^p is written conditionally: E_4 contains no element b anymore and b^p is not the p-th power of anything) can only be killed by the differential $d^0_{2p+1}\colon E^{0,2p}_{2p+1} \to E^{2p+1,0}_{2p+1}$. Further, b^p may not be sent to any polynomial of c and d (which are all "occupied"). Its image $d_{2p+1}b^p$ originates from an element

$$e \in E_2^{2p+1,0} = H^{2p+1}(K(\mathbf{Z}_p, 2); \mathbf{Z}_p)$$

which represents a new multiplicative generator.

The last difficulty we must overcome in calculating the cohomology of $K(\mathbf{Z}_p, 2)$ up to dimension $2p^2$, is to show that d_{2p-2} sends $b^{p-1}d$ to zero. (This is not trivial. It might be sent either to $c^{p-1}da$ or to ea, what would imply either $d_2(c^{p-1}da) = c^p d = 0$ or $d_2(ea) = ce = 0$; none of these does contradict to anything so far.)

Actually $d_{2p-2}(b^{p-1}d) = 0$ as it will be shown later. Now we examine the consequences.

Once $d_{2p-2}(b^{p-1}d) = 0$, then $d_{2p-1}(b^{p-1}d)$ is not zero but represents an element which originates from a further multiplicative generator

$$f \in E_2^{2p+2,0} = H^{2p+2}(K(\mathbf{Z}_p, 2); \mathbf{Z}_p).$$

The element $ae \in E_2^{2p+1,1}$ is carried by d_2 into ce (hence $ce \neq 0$, as $ce = 0$ implies that ae remains in E_∞). Similarly $d_3(be) = de \neq 0$. The column above e contains $ab^s e$ and $b^s e$. We have $d_2(ab^s e) = b^s ce \neq 0$ and $d_3(b^s e) = sb^{s-1}de \neq 0$ if s does not divide p. Therefore

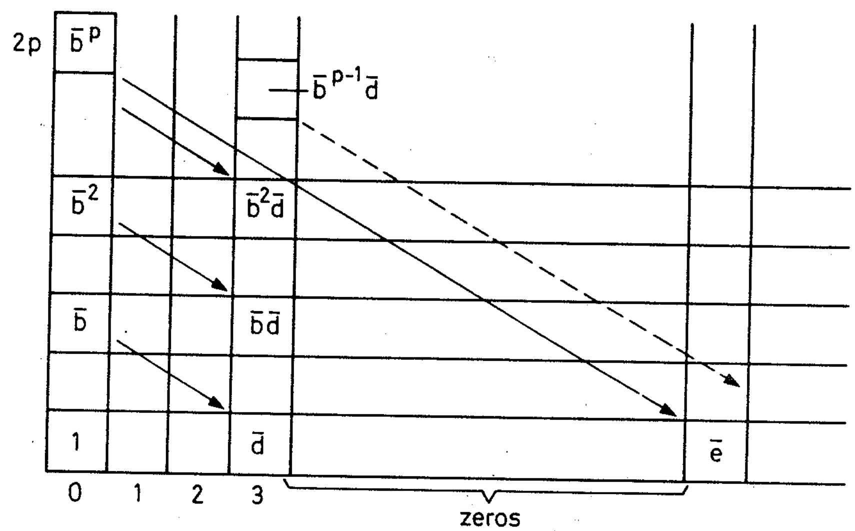
this column in E_4 contains the elements of type $b^{kp}e$ and only them. Similarly in the column over f there remain the elements $b^{kp}f$. Obviously $d_{2p+1}(b^{kp}) = kb^{(k-1)p}e$ and $d_{2p-1}(b^{kp-1}d) = d_{2p-1}(b^{p-1}\cdot d\cdot b^{(k-1)p}) = b^{(k-1)p}f$. Therefore all elements in the first $2p+2$ columns of E_4 are killed by the differentials, up to b^{p^2} and $b^{(p-1)p}e$ which remain. They go into new generators (of dimensions $2p^2$ and $2p^2+1$ (in row zero which contains no generators or relations under these dimensions (i. e. there are all possible polynomials of c, d, e and f while e^2 and d^2 are equal to zero).

By using the same argument we can show that the multiplicative generators of $H^*(K(\mathbf{Z}_p, 2); \mathbf{Z}_p)$ are in the dimensions 2, 3; $2p+1$, $2p+2$; $2p^2+1$, $2p^2+2$; ... while $H^*(K(\mathbf{Z}_p, 2); \mathbf{Z}_p)$ is tensor product of a polynomial ring of even-dimensional generators and an exterior algebra of odd-dimensional generators.

The proof is not wholly trivial; even the part we have done contains a gap which will now be filled in.

We have to prove that $d_{2p-2}(b^{p-1}d) = 0$.

Consider the fibration $* \sim E \xrightarrow{K(\mathbf{Z}, 2)} K(\mathbf{Z}, 3)$. The E_2 term of the cohomology spectral sequence is as follows:

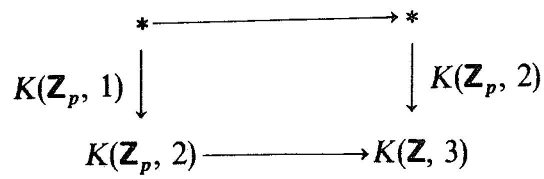


We already know the cohomology of the fibre ($\mathbf{CP}^\infty$). As it can easily be seen $H^*(K(\mathbf{Z}, 3); \mathbf{Z}_p)$ has a single additive generator $\bar{d}$ under the dimension $2p+1$ and $d_i(\bar{b}^{p-1}\bar{d}) = 0$ for any $i \le 2p-3$ (by consideration of dimensions). There exists a mapping φ: $K(\mathbf{Z}_p, 2) \to K(\mathbf{Z}, 3)$ such that $\varphi^*(\bar{d}) = d \in H^3(K(\mathbf{Z}_p, 2); \mathbf{Z}_p)$. Indeed, such a mapping may be constructed by using an element of $H^3(K(\mathbf{Z}_p, 2); \mathbf{Z})$ (as it can be shown, for instance, by considering the integral spectral sequence of the fibration $* \xrightarrow{K(\mathbf{Z}_p, 1)} K(\mathbf{Z}_p, 2)$). We have $H^3(K(\mathbf{Z}_p, 2); \mathbf{Z}) = \mathbf{Z}_p$, hence $d \in H^3(K(\mathbf{Z}_p, 2); \mathbf{Z}_p)$ is an integral element, i. e. it is contained in the image of the homomorphism

$$\rho_p\colon H^3(K(\mathbf{Z}_p, 2); \mathbf{Z}) \to H^3(K(\mathbf{Z}_p, 2); \mathbf{Z}_p).$$

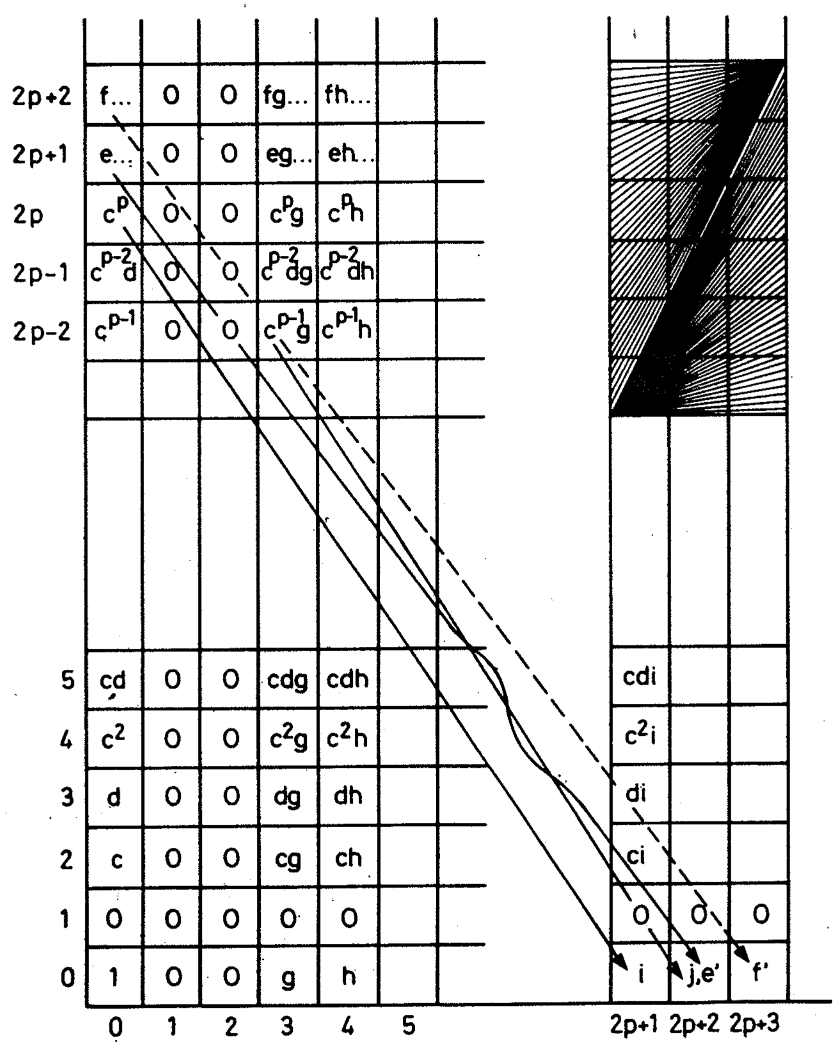
We construct a mapping $K(\mathbf{Z}_p, 2) \to K(\mathbf{Z}, 3)$ according to any pre-image of d in $H^3(K(\mathbf{Z}_p, 2); \mathbf{Z})$ of d. It has the required properties.

The mapping φ: $K(\mathbf{Z}_p, 2) \to K(\mathbf{Z}, 3)$ induces a mapping of the loop spaces and also of the Serre fibrations



The homomorphism induced in the spectral sequences send $\bar{d}$ into d (by construction of φ) and $\bar{b}$ into b ($\bar{b}$ is sent to such an element b' that $d_2 b' = d$; hence $b' = \bar{b}$); $\bar{b}^{p-1}\bar{d}$ is sent into $b^{p-1}d$ and, finally, $d_{2p-2}(\bar{b}^{p-1}\bar{d}) = 0$ is sent into $d_{2p-2}(b^{p-1}d)$, hence $d_{2p-2}(b^{p-1}d) = 0$.

Remark. We emphasize that in the case considered, we have $p \neq 2$. It will be recommended to the reader to examine $H^*(K(\mathbf{Z}_2, 2); \mathbf{Z}_2)$ and see the situation become rather complicated, as compared to the case $p \neq 2$, because $d^2 \neq 0$ and even the desirable relation $d^2 = c^3$ is not valid.



Let us now consider the ring $H^*(K(\mathbf{Z}_p, 3); \mathbf{Z}_p)$. Consider the fibration

$$* \sim E \xrightarrow{K(\mathbf{Z}_p, 2)} K(\mathbf{Z}_p, 3).$$

In the first four columns d_2 is trivial by consideration of dimensions. Obviously $d_3(c) \neq 0$; set $d_3(c) = g$. Then $d_3(c^k) = kc^{k-1}g$, and as far as $k < g$, all elements of type $c^{k-1}g$ in the third column are "covered" by the differential d_3 and will not go into E_2. Since $d_3(d) = 0$, we have $d_4(d) \neq 0$; set $d_4(d) = h$.

By the same consideration as above the elements of the first $2p$ columns below c^p are killed by the third and fourth differentials, and the bottom row contains no generators but g and h and no relations but $g^2 = 0$.

Again the first anomaly appears when $d_3(c^p) = pc^{p-1}g = c$ and the element $c^{p-1}g$ is not killed. The generator c^p is taken by d_{2p+1} into a new generator

$$i \in H^{2p+1}(K(\mathbf{Z}_p, 3); \mathbf{Z}_p),$$

and $c^{p-1}g$ is taken into a new generator j of $H^{2p+2}(K(\mathbf{Z}_p, 3); \mathbf{Z}_p)$.

The elements e and f are also transgressive and so we obtain in $H^*(K(\mathbf{Z}_p, 3); \mathbf{Z}_p)$ six multiplicative generators of dimensions $3, 4; 2p+1, 2p+2, 2p+2, 2p+3$. As above, we can show that there are no more generators under the dimension $2p^2+1$ and no relations except those of skew-commutativity (i. e.

$$H^*(K(\mathbf{Z}_p, 3); \mathbf{Z}_p) \text{ and } \mathbf{Z}_p[h, j, e'] \otimes \Lambda\,(g, i, f')$$

are isomorphic algebras up to the dimension $2p^2$ included).

Further we consider the spectral sequences of the fibrations

$$* \xrightarrow{K(\mathbf{Z}_p, 3)} K(\mathbf{Z}_p, 4),$$

$$* \xrightarrow{K(\mathbf{Z}_p, 4)} K(\mathbf{Z}_p, 5),$$

etc. The generators we find are transgressive; they go over into $H^*(K(\mathbf{Z}_p, 4); \mathbf{Z}_p)$ and then to $H^*(K(\mathbf{Z}_p, 5); \mathbf{Z}_p)$, etc. In $H^*(K(\mathbf{Z}_p, n); \mathbf{Z}_p)$ they become generators of dimensions $n, n+1, n+2p-2, n+2p-1, n+2p-1, n+2p$. No other generators exist under the dimension $n+4p-4$. (As a rule, new generators come under transgression from the p-th power of even-dimensional generators in the fibre. Like $H^*(K(\mathbf{Z}_p, 1); \mathbf{Z}_p)$, $H^*(K(\mathbf{Z}_p, 2); \mathbf{Z}_p)$ had two-dimensional generators. Their p-th powers had dimension $2p$. That led to the arising of generators in the dimensions near to $2p$. Now the first even-dimensional generator in $H^*(K(\mathbf{Z}_p, 3); \mathbf{Z}_p)$ has dimension 4, so its p-th power has dimension $4p$. Still larger are the dimensions of the generators in $H^*(K(\mathbf{Z}_p, n); \mathbf{Z}_p)$. Thus there are no new generators of dimension $< n+4p-4$.

It should be noted that if n is large enough ($n \geq 4p-4$) then the products of these generators have dimension $\geq n+4p-4$, therefore the cohomology ring of $K(\mathbf{Z}_p, n)$ mod p not only have no further generators but even there are no further elements, up to the dimension $n+4p-4$.

Thus we have the following theorem (with some gaps which the reader will bridge).

Theorem. If $n \geq 3$, the algebra $H^*(K(\mathbf{Z}_p, n); \mathbf{Z}_p)$ up to the dimension $n+4p-4$ is isomorphic to an algebra with generators in the dimensions $n, n+1, n+2p-2, n+2p-1, n+2p-1, n+2p$ and without relations except those of skew-commutativity.

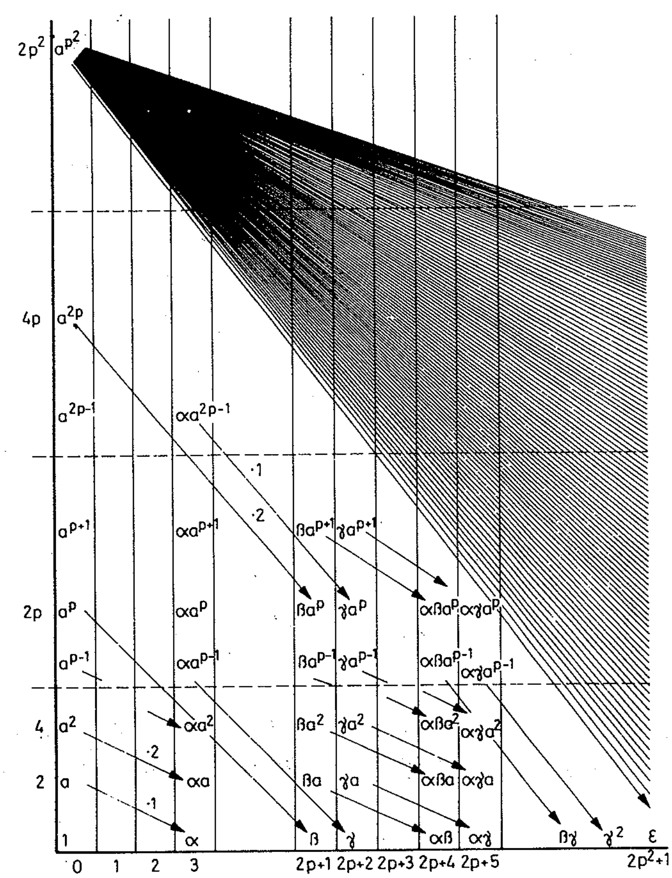
If $n \geq 4p-4$ then

$$H^q(K(\mathbf{Z}_p, n); \mathbf{Z}_p) = \begin{cases} \mathbf{Z}_p & \text{for } q=0,\ n,\ n+1,\ n+2p-2,\ n+2p, \\ \mathbf{Z}_p \oplus \mathbf{Z}_p & \text{for } q=n+2p-1, \\ 0 & \text{for all other } q<n+4p-4. \end{cases}$$

The further calculation of $H^*(K(\mathbf{Z}_p, n); \mathbf{Z}_p)$ may be found, for example, in the article: Postnikov M. M. On the H. Cartan Theorem. Uspechi Mat. Nauk 1966. V. 21. No. 4. pp. 35–46.

Computing $H^*(K(Z, n); Z_p)$

The computations are similar to those above. As we know, $K(\mathbf{Z}, 2) = \mathbf{CP}^\infty$ thus $H^*(K(\mathbf{Z}, 2); \mathbf{Z}_p) = \mathbf{Z}_p[a]$ where $\deg a = 2$. Consider the cohomology spectral sequence mod p of the fibration $* \xrightarrow{K(\mathbf{Z}, 2)} K(\mathbf{Z}, 3)$ (already examined in some extent). The E_2 term and the action of the differentials is shown on the following diagram.



We obtain that $H^*(K(\mathbf{Z}, 3); \mathbf{Z}_p)$ is in the dimensions $\leq 2p^2$ isomorphic to the algebra $\mathbf{Z}_p[\gamma] \otimes \wedge(\alpha, \beta)$ where $\deg \alpha = 3$, $\deg \beta = 2p+1$ and $\deg \gamma = 2p+2$.

We note that the lack of any generator in the dimension 2 will make the work easier as there will be no further generators arising in the dimensions near to $2p$. It is left to the reader to examine the spectral sequences of the fibrations $* \xrightarrow{K(\mathbf{Z}, 3)} K(\mathbf{Z}, 4)$, $* \xrightarrow{K(\mathbf{Z}, 4)} K(\mathbf{Z}, 5)$, etc.

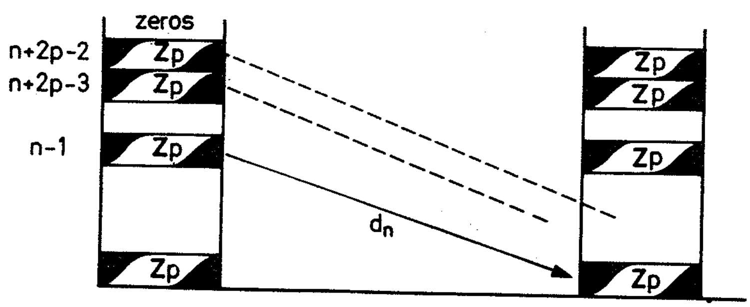
The final result is the next.

Theorem. For $n \geq 3$ the algebra $H^*(K(\mathbf{Z}, n); \mathbf{Z}_p)$ in the dimensions $< n+4p-4$ is isomorphic to the algebra with generators in the dimensions $n, n+2p-2, n+2p-1$ and without any relations except those of skew-commutativity.

For $n \geq 4p-4$

$$H^q(K(\mathbf{Z}, n); \mathbf{Z}_p) = \begin{cases} \mathbf{Z}_p & \text{for } q = 0, n, n+2p-2, n+2p-1, \\ 0 & \text{for all other } q < n+4p-4. \end{cases}$$

Now we apply the results to the homotopy groups of spheres. Let p be small as compared to n and not equal to 2. Consider the fibration $S^n|_{n+1} \xrightarrow{K(\mathbf{Z}, n-1)} S^n$.



The spectral sequence is considered over $\mathbf{Z}_p$. Since $\tau = d_n$ is known to be an isomorphism of $E_n^{0, n-1}$ to $E_n^{n, 0}$, we have

$$H^i(S^n|_{n+1}; \mathbf{Z}_p) = \begin{cases} 0 & \text{for } 0 < i < n+2p-3 \text{ and } n+2p-2 < i < n+4p-4, \\ \mathbf{Z}_p & \text{for } i = 0,\ i = n+2p-3 \text{ and } i = n+2p-2, \\ \text{something} & \text{for } i > n+4p-4. \end{cases}$$

In particular, $H_{n+1}(S^n|_{n+1}; \mathbf{Z}_p) = 0$, i. e. $\pi_{n+1}(S^n)$ contain no summands whose order is divisible by p (i. e. the p-component of $\pi_{n+1}(S^n)$ is zero).

Consider the following killing space $S^n|_{n+2}$:

$$S^n|_{n+2} \xrightarrow{K(\pi, n)} S^n|_{n+1}$$

where $\pi = \pi_{n+1}(S^n|_{n+1})$. As we have just shown the group π contains no elements of orders divisible by p, i. e. $H^*(K(\pi, n); \mathbf{Z}_p) = 0$ for $i > 0$. Hence $H^*(S^n|_{n+2}; \mathbf{Z}_p) \cong$ $\cong H^*(S^n|_{n+1}; \mathbf{Z}_p)$.

We obtain that $\pi_{n+2}(S^n)$ has no element whose order is divisible by p. Indeed, the vanishing of the cohomology groups mod p implies that the homology groups mod p are zero, too, because $H^k(X; \mathbf{Z}_p) = H^k(X) \otimes \mathbf{Z}_p \oplus \mathrm{Tor}\,(H^{k+1}(X); \mathbf{Z}_p)$.

We may go on in the same way as long as dimension $n+2p-3$ will not have been reached. Thus we obtain the following result.

Theorem. $\pi_{n+i}(S^n) \otimes \mathbf{Z}_p = 0$ for any prime p and any $0<i<2p-3$ i. e. the p-components of these homotopy groups are zero; here p is assumed to be small compared to n (more exactly, $n-1>2p-3$).

Assume now that $n>2p-3$. In this case $\pi_{n+2p-3}(S^n) \otimes \mathbf{Z}_p = \mathbf{Z}_p$. Indeed, $H^{n+2p-3}(S^n|_{n+2p-3}; \mathbf{Z}_p) = \mathbf{Z}_p$ where this lonely $\mathbf{Z}_p$ came from $H^{n+2p-2}(S^n|_{n+2p-3}; \mathbf{Z})$, i. e. $H_{n+2p-3}(S^n|_{n+2p-3}; \mathbf{Z}_p) = \mathbf{Z}_p$. Thus the theorem may be completed by the following statement: $\pi_{n+2p-3}(S^n) \otimes \mathbf{Z}_p = \mathbf{Z}_p$, for any prime number $p \neq 2$ which is small as compared to n (namely, $n-1>2p-3$).

Then we have $\pi_{n+2p-3}(S^n) = \mathbf{Z}_{p^h} \oplus \ldots$ where the last summand is a group whose order is not divisible by p. It will be proved in the sequel that $h=1$, i. e.

$$\pi_{n+2p-3}(S^n) = \mathbf{Z}_p \oplus \ldots$$

Though it was only proved for $p \neq 2$, actually it is true for $p=2$, too, because $\pi_{n+2\cdot 2-3}(S^n) = \pi_{n+1}(S^n) = \mathbf{Z}_2$ $(n \geq 3)$. Thus the relation holds for π_{n+2p-3} for every prime p.

Compare this result with the table of homotopy groups at the end of the book. We read that $\pi_{n+3}(S^n) = \mathbf{Z}_{24}$ for $n \geq 5$. Let $p=3$ and assume n to be large (actually $n>4$ is enough). Then, by the theorem, $\pi_{n+1}(S^n)$ and $\pi_{n+2}(S^n)$ have no 3-component, as also seen on the table, since $\pi_{n+1}(S^n) = \pi_{n+2}(S^n) = \mathbf{Z}_2$ for $n \geq 3$. Now $2p-3 = 3$, hence $\pi_{n+3}(S^n)$ must have a direct summand $\mathbf{Z}_3$. And really we have $\pi_{n+3}(S^n) = \mathbf{Z}_{24} = \mathbf{Z}_3 \oplus \mathbf{Z}_{2^3}$.

Let us still consider a further example, for instance $p=5$. Then $2p-3 = 7$ i. e. for those n sufficiently large as compared to 5 we have the equality $\pi_{n+i}(S^n) \otimes \mathbf{Z}_5 = 0$ for $i<7$. On the table we read that the groups

$$\pi_{n+1}(S^n) = \mathbf{Z}_2,\ n \geq 3; \quad \pi_{n+2}(S^n) = \mathbf{Z}_2,\ n \geq 3;$$

$$\pi_{n+3}(S^n) = \mathbf{Z}_3 \oplus \mathbf{Z}_{2^3},\ n \geq 5; \quad \pi_{n+4}(S^n) = 0,\ n \geq 6;$$

$$\pi_{n+5}(S^n) = 0,\ n \geq 7; \quad \pi_{n+6}(S^n) = \mathbf{Z}_2,\ n \geq 5.$$

have no 5-components, while $\pi_{n+7}(S^n) = \mathbf{Z}_{240} = \mathbf{Z}_5 \oplus \mathbf{Z}_{48}$ for $n \geq 9$.